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CONTENTS

Mathematics. — *Invarianten der Integranden vielfacher Integrale in der Variationsrechnung.* I. By Prof. L. KOSCHMIEDER in Brünn. (Communicated by Prof. R. WEITZENBÖCK).

(Communicated at the meeting of October 29, 1927).

Durch das folgende hoffe ich erstens den Gegenstand zweier meiner vorhergehenden Mitteilungen [1] [2]) weiter zu fördern: die Ermittelung von Invarianten der Funktion F (14) im Zusammenhange mit der *ersten* Variation des Integrals I (15), dessen Integrand die Grösse F ist als Funktion von $n+1$ Abhängigen und deren ersten Ableitungen nach n Unabhängigen. Ich werde hier auf die damals erzielten Ergebnise zurückgreifen und die dort verwendeten Bezeichnungen benutzen, soweit das zweckmässig ist.

Zweitens aber und hauptsächlich will ich unter dem gleichen Gesichtspunkte die *zweite* Variation in die Betrachtung einbeziehen. Dabei folge ich dem Wege, den A. L. UNDERHILL [3]) bei *einfachen* Integralen eingeschlagen hat; es scheint mir sehr bemerkenswert, dass die von UNDERHILL für $n=1$ durchgeführte, schon in diesem Sonderfalle ziemlich verwickelte Rechnung sich, wie ich zeigen werde, auf *n-fache* Integrale (15) verallgemeinern lässt.

Ich habe a.a.O. [1]) [2]) invariantentheoretische Begriffe nur in dem Umfange herangezogen, wie es zur Aufstellung zweier besonderer Invarianten, der „mittleren extremalen Flächenkrümmung" und des „Rauminhaltes", erforderlich war. Im Hinblick auf die folgenden sowohl wie auch auf künftige Untersuchungen ist es angebracht, mit einer allgemeinen invariantentheoretischen Erörterung zu beginnen (I). Diese kann man mit Rücksicht auf die für $n=1$ von UNDERHILL [4]) gegebenen einschlägigen Entwickelungen kurz halten, auch wenn man sie soweit wie möglich sogleich auf eine Funktion G (3) bezieht, die von umfassenderer Art ist als F. Den Anfang (§ 1) bilden Bemerkungen über Punkttransformation. Es ist vorteilhaft, ausser der ersten Reihe von Veränderlichen y eine zweite (§ 2) in Betracht zu ziehen, deren Mitglieder η sich kogredient zu den Werten y transformieren. Wie man unter geeigneten Umständen von einer die Grössen η enthaltenden Invariante zu einer andern übergehen kann, in der nur Grössen y auftreten, zeigt § 3 im Sonderfalle (14). In § 4 kommt die Wirkung des Differentiations- und Variations-

[1]) Math. Zeitschr. **24** (1926), S. 181—190.

[2]) Math. Annalen **94** (1925), S. 252—261.

[3]) Invariants of the Function $F(x, y, x', y')$ in the Calculus of Variations, Trans. Amer. Math. Soc. **9** (1908), S. 316—338.

[4]) A. a. O. [3]), S. 317—326, 331.

verfahrens auf die Invarianz zur Sprache. Entsprechende Fragen bei der Parametertransformation sind Gegenstand des § 5.

Diesem allgemeinen Teile folgt die eingangs angekündigte Anwendung (II, III) auf den besonderen Fall (14), (15). Bei der Betrachtung der ersten Variation (II, 1) ergibt sich (§ 6) die a.a.O. [1], [2]) nicht erörterte Invarianz der Transversalität und der WEIERSTRASSschen E-Funktion; diejenige des durch sein Verschwinden die Extremalen des Integrals I kennzeichnenden Ausdrucks W wird einfacher hergeleitet als a.a.O. [1]), § 3. Um aus W die absolute Invariante S zu erhalten, zieht man (§ 7) die Grössen $\Phi_{\alpha\beta}$ heran; es wird die Übereinstimmung ihrer Determinante Φ mit der von TH. DE DONDER [5]) eingeführten Funktion F_1 dargetan und so die Beziehung zu einer von ihm a.a.O. [5]) angegebenen Invariante hergestellt. In § 8 weise ich auf das Integral I als Anlass zu einer Massbestimmung im $n+1$-stufigen Raume hin.

Der nächste Abschnitt (II, 2), der die zweite Variation betrifft, ist das Kernstück dieser Abhandlung. Ich habe die auf diesen Gegenstand bezüglichen, im bisherigen Schrifttum vorliegenden Ergebnisse in einer kürzlich erschienenen Arbeit [6]) aufgezählt; dort leite ich ferner die Formel für $\delta^2 I$ in Bezug auf eine Extremale her. *Ohne* diese Einschränkung scheint diese Rechnung bei *beliebiger* [7]) Variation noch nicht durchgeführt zu sein. Diese Lücke möchte ich hier ausfüllen: Mit Benutzung der a.a.O. [6]) beschafften Rechenhilfsmittel bringe ich in § 9 die Grösse $\delta^2 F$ auf solche Gestalt, wie sie im Falle *einfacher* Integrale durch die Transformation von UNDERHILL [8]) zustandekommt. Diese ist die Verallgemeinerung der bekannten Transformation von WEIERSTRASS [9]), insofern als sie sich nicht wie diese nur auf eine Extremale, sondern auf eine beliebige Kurve bezieht.

Nunmehr werden (§ 10—§ 13) die mit der zweiten Variation zusammenhängenden Invarianten Ψ, U, Ψ_0, U_0 gewonnen, die für $n=1$ bez. in die von UNDERHILL [10]) gefundenen Invarianten Φ, K, Φ_0, K_0 übergehen. In § 10 spalten wir von dem Ausdrucke $\delta^2 F$ die Invariante Ψ ab. Sie enthält die Variationen δx_i; eine von diesen Veränderlichen der zweiten Reihe freie Invariante U (§ 11) ergibt sich aus Ψ nach dem Verfahren des § 3. Zur Verallgemeinerung der Invariante K von UNDERHILL gelangt auch DE DONDER, aber auf anderem als dem eben geschilderten Wege [11]).

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5) C. R. Acad. sc. Paris **155** (1912), S. 1005.

6) Revista Matemática Hispano-Americana (2) **1** (1926), S. 129—146.

7) Unter Zugrundelegung einer *Normal*variation ist die zweite Variation eines Doppelintegrals von A. KNESER berechnet: Lehrbuch der Variationsrechnung (2. Aufl. 1925), S. 339—348.

8) A. a. O. [3]), § 6, § 7.

9) Vgl. z. B. O. BOLZA, Vorlesungen über Variationsrechnung (1909), S. 224—227.

10) A. a. O. [3]), §§ 8, 9, 12, 13.

11) A. a. O. [5]). — Zur Herleitung des Zusammenhanges der dort angegebenen Determinante von Linearformen mit unserem Ausdrucke (110) scheint längere Rechnung erforderlich.

Ψ, U sind *Punkt*invarianten. Die von mir weiterhin ermittelten, bei Punkt- *und* Parametertransformation invarianten Funktionen Ψ_0, U_0 dürften hier zum ersten Male angegeben sein. Man erhält die erstere (§ 12), indem man die Invariante S variiert und von δS invariante Bestandteile absprengt. Aus Ψ_0 geht schliesslich (§ 13) die nur Veränderliche der ersten Reihe enthaltende Invariante U_0 wiederum dadurch hervor, dass man die in Ψ_0 auftretenden Variationen δx_i nach § 3, § 5 durch kogrediente Grössen ersetzt. Mit Hilfe von U bzw. U_0 lässt sich bei einer Extremale die zweite Variation $\delta^2 I$ in einfacher Weise ausdrücken.

Im letzten Teile (III) befasse ich mich mit einer etwas anders gerichteten Fragestellung, die auf M. FUJIWARA zurückgeht. Dieser hat im Falle $n = 2$, indem er die BELTRAMIschen Differentiatoren in Bezug auf eine während der Rechnung auftretende invariante quadratische Differentialform benutzte, für eine festberandete Extremale eine Zerlegung der Grösse $\delta^2 I$ in drei parameterinvariante Integrale gefunden [12]; ich habe a. a. O. [6] dieses Ergebnis, gleichfalls unter der Annahme $W = 0$, auf n-fache Integrale übertragen. Mit Hilfe mehrerer dort bewiesener Formeln leite ich hier allgemeiner bei einer *beliebigen* Überfläche eine entsprechende Zerspaltung der zweiten Variation in parameterinvariante Summanden her.

I. *Invariantentheoretisches.*

Die Operationen, die uns zuerst beschäftigen, beziehen sich auf die Bestimmungszahlen x_i $(i = 1, 2, \ldots, N)$ eines Punktes in einem Bereiche X_N eines N-stufigen Raumes. Es handelt sich um die Gruppe der analytischen *Punkttransformationen* [13]

$$x_i = \mathfrak{z}_i\,(x'_1, x'_2, \ldots, x'_N), \quad \ldots \ldots \ldots \quad (1)$$

die in dem Bereiche X'_N die von Null verschiedenen Funktionaldeterminanten $\partial(x_1, \ldots, x_N)/\partial(x'_1, \ldots, x'_N) = D$ besitzen und zwischen X'_N und seinem Bilde X_N eine umkehrbar eindeutige Beziehung vermitteln.

In X_N betrachten wir einen Raum X_n von n Stufen $(n < N)$

$$x_i = \varphi_i\,(u_1, u_2, \ldots, u_n); \quad \ldots \ldots \ldots \quad (2)$$

die rechten Seiten in (2) seien analytische Funktionen der in einem Bereiche O_n unabhängig veränderlichen Parameter u_α $(\alpha = 1, 2, \ldots, n)$. Es sei eine Funktion

$$G = G\,(x_i, x_{i,\alpha_1}, \ldots, x_{i,\alpha_1 \ldots \alpha_q}) \quad \ldots \ldots \ldots \quad (3)$$

gegeben, die die Abhängigen (2) und deren Ableitungen [14] nach den

[12]) Tokyō Sūgaku-Buturigakkwai Kizi (2) **6** (1911/12), S. 123—127.

[13]) Vgl. für $N = 2$ BOLZA [9]), S. 343. — Man hat es in (1) und durchweg im folgenden mit reellen Funktionen reeller Veränderlicher zu tun.

[14]) Wir bezeichnen diese wie a. a. O. [15]) durch griechische Zeiger. Wo es ohne Einbusse an Deutlichkeit möglich ist, sparen wir lateinische und griechische Zeiger und schreiben eine $\mathfrak{r}$-te Ableitung kurz $\mathfrak{d}^{\mathfrak{r}} x$.

Unabhängigen u_α bis zur q-ten Ordnung [15]) enthält und in ihren Argumenten analytisch ist. Die *Grundfunktion* G dient als Integrand des über O_n zu erstreckenden Grundintegrals

$$J = \int^{(n)} G du, \quad du = du_1\, du_2 \ldots du_n \quad \ldots \ldots \quad (4)$$

Die zweite Gruppe, die uns hier angeht, besteht aus den analytischen *Parametertransformationen* [16])

$$u_\alpha = \mathfrak{u}_\alpha(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n), \quad \ldots \ldots \quad (5)$$

die in dem Bereiche $\bar{O}_n$ mit positiven Funktionaldeterminanten $\mathfrak{D}$ [1]), (12) versehen sind und diesen ein-eindeutig auf O_n abbilden. Die Funktion G verwandle sich unter der Wirkung von (5) nach der Formel

$$G(x, \bar{\mathfrak{d}}x, \ldots, \bar{\mathfrak{d}}^q x) = \mathfrak{D} G(x, \mathfrak{d}x, \ldots, \mathfrak{d}^q x) \quad ^{17)}; \quad \ldots \quad (6)$$

dann ist das Integral (4) parameterinvariant.

§ 1. *Erweiterte Punkttransformationen.*

Die Gruppe der Transformationen (1) *erweitern* wir wie folgt. Wir sehen die aus (2) zu berechnenden Grössen $\mathfrak{d}^{\mathfrak{r}} x\ (\mathfrak{r} = 1, 2, \ldots, r)$ als neue zu den x hinzutretende Veränderliche an, die sich unter dem Einflusse von (1) nach bestimmten Gesetzen verwandeln:

$$\left.\begin{aligned} x_{i,\alpha} &= \frac{\partial x_i}{\partial x'_k}\, x'_{k,\alpha}, \\ x_{i,\alpha\beta} &= \frac{\partial^2 x_i}{\partial x'_k\, \partial x'_l}\, x'_{k,\alpha}\, x'_{l,\beta} + \frac{\partial x_i}{\partial x'_k}\, x'_{k,\alpha\beta}, \\ &\ldots\ldots\ldots\ldots\ldots \quad ^{18)} \end{aligned}\right\} \quad \ldots \quad (7)$$

Das Ergebnis der Einsetzung der Werte (1), (7) in G (3) werde mit G' bezeichnet,

$$G(x, \mathfrak{d}x, \ldots, \mathfrak{d}^q x) = G'(x', \mathfrak{d}x', \ldots, \mathfrak{d}^q x') \quad \ldots \quad (8)$$

Den Grössen (1), (7) fügen wir die Funktion G und ihre Teilableitungen nach irgend welchen ihrer Argumente (3) bis zur p-ten Ordnung $G^{(\mathfrak{p})}\ (\mathfrak{p} = 1, 2, \ldots, p)$ als weitere Veränderliche hinzu, deren Verwandlungseigenschaften bei dem Wechsel (1) sich durch geeignete Differentiation von (8) ergeben, z.B.

$$\frac{\partial G}{\partial x_{i,\alpha_1 \ldots \alpha_q}} = \frac{\partial G'}{\partial x'_{k,\alpha_1 \ldots \alpha_q}}\, \frac{\partial x'_k}{\partial x_i}, \quad \ldots \ldots \quad (9)$$

$$\ldots\ldots\ldots\ldots$$

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[15]) Besondere von solchen Integranden herrührende Invarianten habe ich unlängst angegeben, Math. Zeitschr. 25 (1926), S. 74—86.

[16]) Vgl. für $n = 2$ Bolza [9]), S. 664.

[17]) $\bar{\mathfrak{d}}x, \ldots$ bedeuten wie a. a. O. [15]), S. 77, die Ableitungen der x_i nach den $\bar{u}_\alpha$.

[18]) Lateinische Zeiger laufen von 1 bis N, griechische von 1 bis n. Ueber die Fort-

Die Erweiterungen (7), (8), (9) der Transformationen (1) zusammen mit diesen selbst bilden dann diejenige unendliche kontinuierliche Gruppe $\mathfrak{Y}$, auf welche es hier ankommt.

Es liege nun irgend eine Funktion $g(x, \mathfrak{d}x, \ldots, \mathfrak{d}^r x, G, G^{(1)}, \ldots, G^{(p)})$ vor. Mit g' bezeichnen wir kurz den Ausdruck, der aus den Veränderlichen $x', \ldots, G'^{(p)}$ nach demselben Gesetze gebildet wird wie g aus $x, \ldots, G^{(p)}$,

$$g' = g(x', \mathfrak{d}x', \ldots, \mathfrak{d}^r x', G', G'^{(1)}, \ldots, G'^{(p)}) \quad . \; . \; . \quad (10)$$

Besitzt im besonderen die Funktion g bei allen Transformationen von $\mathfrak{Y}$ die Eigenschaft

$$g' = D^a g, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

so heisst g eine *Invariante* dieser Gruppe vom Gewichte a.

Die Funktion G selbst ist hiernach gemäss (8) eine absolute Invariante, d.h. eine solche vom Gewichte 0.

§ 2. *Einbeziehung kogredienter Veränderlicher.*

Ausser der ersten Reihe Y der von (3) herrührenden Veränderlichen des § 1 tritt in der Variationsrechnung des Integrals (4) eine zweite Reihe η von Veränderlichen $\xi_{i;\alpha_1 \ldots \alpha_\tau}$ auf, die sich unter der Wirkung von (1) kogredient zu den die Reihe y bildenden Grössen $x_{i,\alpha_1 \ldots \alpha_\tau}$ transformieren,

$$\left.\begin{aligned}
\xi_{i;\alpha} &= \frac{\partial x_i}{\partial x'_k}\, \xi'_{k;\alpha'} \\
\xi_{i;\alpha\beta} &= \frac{\partial^2 x_i}{\partial x'_k\, \partial x'_l}\, \xi'_{k;\alpha}\, \xi'_{l;\beta} + \frac{\partial x_i}{\partial x'_k}\, \xi'_{k;\alpha\beta}, \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad . \; . \; . \; . \quad (12)$$

Indem man die Operationen von $\mathfrak{Y}$ um die Transformationen (12) vermehrt, erweitert man $\mathfrak{Y}$ zu einer neuen Gruppe H. Eine Funktion h der Veränderlichen der beiden Reihen Y und η nennen wir eine Invariante vom Gewichte a gegenüber H, wenn bei allen Transformationen von H

$$h(x'_i, x'_{i,\alpha'}, \ldots, \xi'_{i;\alpha'}, \xi'_{i;\alpha\beta'}, \ldots) = D^a h(x_i, x_{i,\alpha'}, \ldots, \xi_{i;\alpha'}, \xi_{i;\alpha\beta'}, \ldots)$$

oder nach Art der Bezeichnung (11) kurz

$$h' = D^a h \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

ist.

§ 3. *Übergang von einer Invariante h zu einer Invariante g.*

Bei dem im zweiten Teile angewandten Verfahren zur Ermittelung von Invarianten stellen sich mehrfach zunächst solche von der Art h ein.

Da uns hauptsächlich an Invarianten g liegt, so sind Übergangsmöglichkeiten von Invarianten h zu solchen der Art g erwünscht. UNDERHILL[19]) hat darüber im Falle $n=1$, $N=2$, $q=1$ einen Satz aufgestellt; wir beschränken uns hier darauf, diesen auf die Funktion

$$F = G(x_i, x_{i,\alpha}), \quad i = 1, 2, \ldots, n+1 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

als Integranden des statt (4) später ausschliesslich betrachteten Integrals

$$I = \int^{(n)} F\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

zu verallgemeinern. Es handelt sich also bei (14) in den Bezeichnungen (3) um den Sonderfall der Grundfunktion, in dem n beliebig, $N=n+1$ und $q=1$ ist.

Wir benutzen die von J. RADON[20]) und G. VIVANTI[21]) aus der Parameterinvarianz des Integrals I [vgl. (6)] gezogene Folgerung, dass die Grössen $x_{i,\alpha}$ in den Integranden F nur eingehen verbunden zu den Determinanten θ_k [22]), die man bis auf den Faktor $(-1)^{k+1}$ aus der Funktionalmatrix $\| x_{i,\alpha} \|$ dadurch erhält, dass man in ihr die k-te Zeile (der $x_{k,\alpha}$) streicht. Genauer ist

$$F = \Gamma(x_i, \theta_i) \quad ^{23}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

eine Funktion der x_i und θ_i, die in den θ_i positiv-homogen von erster Stufe ist; daher besteht die Beziehung

$$\theta_i \frac{\partial F}{\partial \theta_i} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Was die Veränderlichen (12) betrifft, so fügen wir den Grössen $\xi_{i;\alpha}$ solche ξ_i hinzu, die sich bei der Verwandlung (1) kogredient zu den Differentialen dx_i umsetzen, also nach den Formeln

$$\xi_i = \frac{\partial x_i}{\partial x'_k} \xi'_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Nach diesen Vorbemerkungen beweisen wir folgenden

Satz 1. Aus einer Invariante h vom Gewichte a, die von den Veränderlichen der zweiten Reihe nur die ξ_i enthält und in diesen homogen von b-ter Stufe ist, erhält man eine Invariante g vom Gewichte $a+b$, wenn man die ξ_i bezüglich durch die Grössen $\partial F / \partial \theta_i$ ersetzt. [24])

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lassung des Summenzeichens treffen wir bei beiden Arten von Zeigern die in der Tensorrechnung übliche Vereinbarung; Ausnahmen von dieser machen wir kenntlich.

[19]) A. a. O.[3]), S. 323.

[20]) Monatsh. f. Math. u. Phys. **22** (1911), S. 55.

[21]) Rend. Circ. mat. Palermo **33** (1912), S. 271.

[22]) Wir schreiben θ_k, nicht wie a. a. O.[1]) (S. 181) Δ_k, weil wir den Buchstaben Δ unten (III) in anderem Sinne verwenden.

[23]) Es hat kein Bedenken, das Zeichen F auch nach Einführung der Argumente θ_i beizubehalten; man verstehe $\partial F / \partial \theta_i = \partial \Gamma / \partial \theta_i$ u.s.w. Uebrigens bedienen wir uns, wo es angebracht scheint, der $x_{i,\alpha}$; $\partial F / \partial x_{i,\alpha} = \partial G / \partial x_{i,\alpha}$ u.s.w.

[24]) DE DONDERS Schritt von den linearen Formen DM, $\delta x_i . \partial F / \partial x_{i,\alpha}$ zu deren Deter-

Aus der Invarianz (8) der Grundfunktion

$$F' = F \quad \ldots\ldots\ldots\ldots \quad (19)$$

und der Transformation[1]), (19') der Determinanten θ_i

$$D\theta'_k = \frac{\partial x_i}{\partial x'_k}\,\theta_i \quad \ldots\ldots\ldots\ldots \quad (20)$$

ergibt sich nämlich, dass sich die Ausdrücke $\partial F/\partial\theta_i$ so ändern:

$$D\,\frac{\partial F}{\partial\theta_i} = \frac{\partial x_i}{\partial x'_k}\,\frac{\partial F'}{\partial\theta'_k} \quad \ldots\ldots\ldots\ldots \quad (21)$$

Diese Formeln nehmen die Gestalt (18) an, wenn man $D.\partial F/\partial\theta_i = \xi_i$, $\partial F'/\partial\theta'_k = \xi'_k$ setzt. Nun ist nach Voraussetzung

$$h\,(x'_i, x'_{i,\alpha}, \ldots, \xi'_i) = D^a\, h\,(x_i, x_{i,\alpha}, \ldots, \xi_i),$$

sofern nur (18) gilt; daher ist im besonderen

$$h\left(x'_i, x'_{i,\alpha}, \ldots, \frac{\partial F'}{\partial\theta'_i}\right) = D^a\, h\left(x_i, x_{i,\alpha}, \ldots, D\,\frac{\partial F}{\partial\theta_i}\right).$$

Hieraus folgt die Behauptung mit Rücksicht auf die Homogeneität des Ausdrucks h in den ξ_i.

§ 4. *Differentiation und Variation als invarianzerhaltende Verfahren.*

Indem wir zu der allgemeineren Grundfunktion G (3) zurückkehren, unterwerfen wir eine Invariante der *Differentiation*, erstens nach den Parametern u_α. Treten ausser diesen weitere Veränderliche $\varepsilon_{\mathfrak{a}}$ ($\mathfrak{a} = 1, 2, \ldots, \mathfrak{k}$) auf, die unter sich und von den u_α unabhängig sind, so handelt es sich zweitens um das die Ableitung nach den $\varepsilon_{\mathfrak{a}}$ betreffende δ-*Verfahren* in seiner Wirkung auf Funktionen $\psi\,(u_\alpha, \varepsilon_{\mathfrak{a}})$. In der Variationsrechnung[25]) setzt man mit Bezug auf eine Funktion $\tilde{\chi}\,(u_\alpha, \varepsilon_{\mathfrak{a}})$, die für $\varepsilon = 0$[26]) in $\chi\,(u_\alpha)$ übergeht, die $\mathfrak{s}$-te Variation von χ ($\mathfrak{s} = 1, 2, \ldots$)

$$\delta^{\mathfrak{s}}\chi = \delta\,(\delta^{\mathfrak{s}-1}\chi) = \left(\sum_{\mathfrak{a}}^{1,\mathfrak{k}} d\varepsilon_{\mathfrak{a}}\,\frac{\partial}{\partial\varepsilon_{\mathfrak{a}}}\right)^{(\mathfrak{s})}\tilde{\chi}\,\Bigg|^{\varepsilon=0} \quad \ldots\ldots \quad (22)$$

Hier erklären wir, ähnlich wie UNDERHILL[27]) im Falle[3]), das $\mathfrak{s}$-malige δ-Verfahren durch die Formel

$$\delta^{\mathfrak{s}}\psi = \delta\,(\delta^{\mathfrak{s}-1}\psi) = \left(\sum_{\mathfrak{a}}^{1,\mathfrak{k}} d\varepsilon_{\mathfrak{a}}\,\frac{\partial}{\partial\varepsilon_{\mathfrak{a}}}\right)^{(\mathfrak{s})}\psi, \quad \ldots\ldots \quad (23)$$

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minante a. a. O. [11]) [dort Erklärung des Zeichens D, wegen M vgl. u. (78)] kann als Ersetzung der δx_i in der ersten von ihnen durch kogrediente Grössen angesehen werden; denn die Determinanten der Matrix $\| \partial F/\partial x_{k,\alpha} \|$, die aus dieser durch Streichung der i-ten Zeile hervorgehen, haben die Werte $F^{n-1}(-1)^{i-1}\,.\,\partial F/\partial\theta_i$. Letzteres beweist im Falle $n=2$ G. VIVANTI [Elementi del calcolo delle variazioni (1923), S. 171]; bei beliebigem n leitet man es leicht durch eine ähnliche Rechnung her, wie wir sie später zum Beweise von (88) durchführen.

[25]) Vgl. KNESER [7]), S. 5/6, 130.

[26]) Wir schreiben kurz $\varepsilon = 0$ statt $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{\mathfrak{k}} = 0$; ebenso später $u = 0$ statt $u_1 = u_2 = \ldots = u_n = 0$.

[27]) A. a. O. [3]), S. 326.

ausführlicher geschrieben

$$\delta^{\mathfrak{s}}\psi = \sum_{\mathfrak{a}_1,\ldots,\mathfrak{a}_{\mathfrak{s}}}^{1,\mathfrak{k}} d\varepsilon_{\mathfrak{a}_1}\ldots d\varepsilon_{\mathfrak{a}_{\mathfrak{s}}} \frac{\partial^{\mathfrak{s}}\psi}{\partial\varepsilon_{\mathfrak{a}_1}\ldots\partial\varepsilon_{\mathfrak{a}_{\mathfrak{s}}}} \quad \ldots\ldots \quad (24).$$

Der Kürze halber sei es gestattet, den Ausdruck $\delta^{\mathfrak{s}}\psi$ als die $\mathfrak{s}$-te [28]) Variation von ψ zu bezeichnen. Das Zeichen der Variation ist mit dem der Differentiation nach den Parametern u_α und dem der Integration über den unvariierten Bereich O_n vertauschbar.

Man denke die x_i als Funktionen A_i der u_α und der $\varepsilon_{\mathfrak{a}}$ und bilde nach (24) die Variationen $\delta^{\mathfrak{s}}\mathfrak{d}^{\mathfrak{r}}x$; diese transformieren sich nach den Formeln

$$\left.\begin{aligned}
\delta x_i &= \frac{\partial x_i}{\partial x'_k}\,\delta x'_k,\\
\delta x_{i,\alpha} &= \frac{\partial^2 x_i}{\partial x'_k\,\partial x'_l}\,\delta x'_k\,x'_{l,\alpha} + \frac{\partial x_i}{\partial x'_k}\,\delta x'_{k,\alpha},\\
&\ldots\ldots\ldots\ldots\ldots\\
\delta^2 x_i &= \frac{\partial^2 x_i}{\partial x'_k\,\partial x'_l}\,\delta x'_k\,\delta x'_l + \frac{\partial x_i}{\partial x'_k}\,\delta^2 x'_k,\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \ldots\ldots \quad (25)$$

Die Variationen $\delta^{\mathfrak{s}}x_i$ verwandeln sich unter dem Einflusse von (1) kogredient zu den Differentialen $d^{\mathfrak{s}}x_i$. Durch Hinzufügung von (25) zu (1), (7), (8), (9), (12) geht aus der Gruppe H (§ 2) abermals eine neue Ξ hervor. Die Invarianz gegenüber Ξ erklären wir entsprechend (11) und (13); Invarianten der Gruppe Ξ heissen *Punktinvarianten*.

Indem man Ableitungen nach den $\varepsilon_{\mathfrak{a}}$ durch deutsche Zeiger andeutet,

$$x_{i,\mathfrak{a}} = \frac{\partial x_i}{\partial\varepsilon_{\mathfrak{a}}},\quad x_{i,\alpha\mathfrak{a}} = \frac{\partial x_{i,\alpha}}{\partial\varepsilon_{\mathfrak{a}}},\ldots,\quad x_{i,\mathfrak{a}\mathfrak{b}} = \frac{\partial^2 x_i}{\partial\varepsilon_{\mathfrak{a}}\,\partial\varepsilon_{\mathfrak{b}}},\ldots$$

kann man den Inhalt von (25) auch in der Gestalt ausdrücken

$$\left.\begin{aligned}
x_{i,\mathfrak{a}} &= \frac{\partial x_i}{\partial x'_k}\,x'_{k,\mathfrak{a}},\\
x_{i,\alpha\mathfrak{a}} &= \frac{\partial^2 x_i}{\partial x'_k\,\partial x'_l}\,x'_{k,\mathfrak{a}}\,x'_{l,\alpha} + \frac{\partial x_i}{\partial x'_k}\,x'_{k,\alpha\mathfrak{a}},\\
&\ldots\ldots\ldots\ldots\ldots\\
x_{i,\mathfrak{a}\mathfrak{b}} &= \frac{\partial^2 x_i}{\partial x'_k\,\partial x'_l}\,x'_{k,\mathfrak{a}}\,x'_{l,\mathfrak{b}} + \frac{\partial x_i}{\partial x'_k}\,x'_{k,\mathfrak{a}\mathfrak{b}},\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \ldots\ldots \quad (26)$$

Es liege nun eine absolute Invariante $\mathfrak{h}(\delta^{\mathfrak{s}}\mathfrak{d}^{\mathfrak{r}}x)$ vor, $\mathfrak{r}=0,1,\ldots,r$; $\mathfrak{s}=0,1,\ldots,s$; wir bezeichnen ihre Argumente kurz mit z_L, $L=1,2,\ldots$ Auf die Differentiation von $\mathfrak{h}$ beziehen sich

Satz 2. Mit $\mathfrak{h}$ ist auch die Ableitung $\partial\mathfrak{h}/\partial u_\alpha$ absolut punktinvariant.

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[28]) Unter $\delta^{\mathfrak{s}}\psi$ für $\mathfrak{s}=0$ verstehen wir ψ.

Satz 3. Mit $\mathfrak{h}$ ist auch die Variation $\delta\mathfrak{h}$ absolut punktinvariant.

Die Beweise beider Sätze führen wir gleichzeitig. Bei dem Ansatze $x'_i = \Lambda'_i(u_\alpha, \varepsilon_\mathfrak{a})$ stellt die vorausgesetzte Invarianz $\mathfrak{h} = \mathfrak{h}'$ auf Grund von (1), (7), (25) eine Identität in *u und* ε [29] dar; man kann sie etwa nach u_α bzw. $\varepsilon_\mathfrak{a}$ differenzieren. Das Ergebnis

$$\frac{\partial\mathfrak{h}}{\partial u_\alpha} = \frac{\partial\mathfrak{h}'}{\partial u_\alpha}, \quad \ldots \quad (27^a) \qquad\qquad \frac{\partial\mathfrak{h}}{\partial \varepsilon_\mathfrak{a}} = \frac{\partial\mathfrak{h}'}{\partial \varepsilon_\mathfrak{a}} \quad \ldots \quad (27^b)$$

hat ausgeführt die Gestalt

$$\sum_L \frac{\partial\mathfrak{h}}{\partial z_L} z_{L,\alpha} = \sum_L \frac{\partial\mathfrak{h}'}{\partial z'_L} z'_{L,\alpha}, \ldots (28^a) \qquad\qquad \sum_L \frac{\partial\mathfrak{h}}{\partial z_L} z_{L,\mathfrak{a}} = \sum_L \frac{\partial\mathfrak{h}'}{\partial z'_L} z'_{L,\mathfrak{a}} \ldots (28^b)$$

Die Beziehungen (28) sind zunächst Identitäten in u und ε; sie bestehen aber auch, wenn man die Gesamtheit Z' der Ableitungen $\bar{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1,\ldots,\mathfrak{a}_\mathfrak{s}}$ [30] in ihnen durch eine solche $\hat{Z}'$ von *beliebigen* Werten $\hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}}$ [31] ersetzt und zugleich Z durch diejenige Gesamtheit $\hat{Z}$ von Werten $\hat{x}_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}}$, welche mit $\hat{Z}'$ durch die Formeln (1), (7), (26) zusammenhängt. Dabei treten in (28) an Stelle der $\delta^\mathfrak{s} x'_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$ die Grössen

$$\delta^\mathfrak{s} \hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}} = \sum_{\mathfrak{a}_1,\ldots,\mathfrak{a}_\mathfrak{s}}^{1,\mathfrak{k}} d\varepsilon_{\mathfrak{a}_1} \ldots d\varepsilon_{\mathfrak{a}_\mathfrak{s}} \hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}} \quad \ldots \quad (29)$$

und an Stelle der $\delta^\mathfrak{s} x_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$ die nach Art von (29) aus den Mitgliedern von $\hat{Z}$ gebildeten Grössen $\delta^\mathfrak{s} \hat{x}_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$.

Um zu zeigen, dass die Invarianzen (28) auch in diesem allgemeineren Sinne gelten, wählen wir die Λ'_i in der Form der Polynome

$$x'_i = \sum_\tau^{0,R} \sum \sum_\mathfrak{t}^{0,S} \sum \hat{x}'_{i,(\tau_1)\ldots(\tau_n)(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{k})} \frac{u_1^{\tau_1}\ldots u_n^{\tau_n}}{\tau_1!\ldots\tau_n!} \frac{\varepsilon_1^{\mathfrak{t}_1}\ldots\varepsilon_\mathfrak{k}^{\mathfrak{t}_\mathfrak{k}}}{\mathfrak{t}_1!\ldots\mathfrak{t}_\mathfrak{k}!} \quad \ldots \quad (30)$$

Dabei ist die vierte bzw. zweite Summe über die Werte $\mathfrak{t}_\mathfrak{a} \geqq 0$ bzw. $\tau_\alpha \geqq 0$ so zu erstrecken, dass $\mathfrak{t}_1 + \ldots + \mathfrak{t}_\mathfrak{k} = \mathfrak{t}$ bzw. $\tau_1 + \ldots + \tau_n = \tau$ ist; S bzw. R bezeichnet in beiden Fällen (*a*), (*b*) den jeweils grössten Wert des Zeigers $\mathfrak{s}$ bzw. $\mathfrak{r}$ [30]; die Koeffizienten bedeuten die Mitglieder von $\hat{Z}'$, die den zu Z' gehörigen Ableitungen

$$\frac{\partial^{\tau+\mathfrak{t}} x'_i}{\partial u_1^{\tau_1}\ldots\partial u_n^{\tau_n}\,\partial\varepsilon_1^{\mathfrak{t}_1}\ldots\partial\varepsilon_\mathfrak{k}^{\mathfrak{t}_\mathfrak{k}}} = x'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{k})}$$

entsprechen. Der Ansatz (30) liefert in der Tat

$$x'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{k})}\Big|^{u=0,\ \varepsilon=0} = \hat{x}'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{k})};$$

---

29) Diese Folge der Erklärung (23) gibt dem Beweise der Aussage (29) bequeme Form.

30) Jetzt läuft im Falle (*a*) $\mathfrak{r}$ bis $r+1$, im Falle (*b*) $\mathfrak{s}$ bis $s+1$.

31) Wir setzen bei ihnen diejenigen Zeigersymmetrien voraus, welche bei den ihnen entsprechenden Ableitungen statthaben.

er bewirkt, dass Z' in $\hat{Z}'$, also $\delta^{\mathfrak{s}} x'_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$ nach (24), (29) in $\delta^{\mathfrak{s}}\hat{x}'_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$ übergeht. Aus der gemachten Annahme, dass $\hat{Z}$ und $\hat{Z}'$ durch (1), (7), (26) verknüpft sind, folgt dann, dass gleichzeitig aus Z die Gesamtheit $\hat{Z}$ wird und mithin die Grössen $\delta^{\mathfrak{s}} x_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$ wirklich die Werte $\delta^{\mathfrak{s}}\hat{x}_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$ erhalten. Die Gleichungen (27^b) ergeben gemäss (23.1) sogleich den Satz 3

$$\delta\mathfrak{h}' = \delta\mathfrak{h} \quad \ldots\ldots\ldots\ldots \quad (31)$$

§ 5. *Parametertransformationen.*

Bei den Parametertransformationen (5), von denen jetzt kurze Rede sein soll, setzen wir voraus, dass in den Ausdrücken u_α die $\varepsilon_{\mathfrak{a}}$ nicht vorkommen. Die u_α, $\bar{u}_\alpha$ werden nicht variiert; in der Schreibweise des § 4 ist

$$\delta u_\alpha = \delta\bar{u}_\alpha = 0 \quad \ldots\ldots\ldots\ldots \quad (32)$$

Ein Wechsel (5) lässt die x_i ungeändert,

$$x_i = \bar{x}_i \quad \ldots\ldots\ldots\ldots \quad (33)$$

Es sei $x_i = A_i\,(u_\alpha, \varepsilon_{\mathfrak{a}}) = \bar{A}_i\,(\bar{u}_\alpha, \varepsilon_{\mathfrak{a}})$; bedient man sich der Abkürzungen

$$x_{i,\bar{\lambda}} = \frac{\partial x_i}{\partial \bar{u}_\lambda}\,, \quad x_{i,\bar{\lambda}\bar{\mu}} = \frac{\partial^2 x_i}{\partial \bar{u}_\lambda\,\partial \bar{u}_\mu}, \ldots,$$

so lauten die Verwandlungsformeln der Ableitungen $\mathfrak{d}^{\mathfrak{r}} x$

$$\left.\begin{aligned} x_{i,\alpha} &= x_{i,\bar{\lambda}}\,\frac{\partial \bar{u}_\lambda}{\partial u_\alpha}, \\ x_{i,\alpha\beta} &= x_{i,\bar{\lambda}\bar{\mu}}\,\frac{\partial \bar{u}_\lambda}{\partial u_\alpha}\,\frac{\partial \bar{u}_\mu}{\partial u_\beta} + x_{i,\bar{\lambda}}\,\frac{\partial^2 \bar{u}_\lambda}{\partial u_\alpha\,\partial u_\beta}, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots \quad (34)$$

Die durch (5) bedingten Transformationen (33), (34) bilden eine Gruppe $\mathfrak{N}$; man kann sie zu einer neuen $\mathfrak{B}$ *erweitern*, indem man wie in § 2 eine zweite Reihe (oder deren mehrere) von Veränderlichen $\mathfrak{v}_{i;\alpha_1\ldots\alpha_{\mathfrak{r}}}$ [32]) einführt, die sich kogredient zu den $x_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$ d. h. so umsetzen, dass die Formeln (33), (34) gelten, wenn man in ihnen $x_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$, $x_{i,\bar{\alpha}_1\ldots\bar{\alpha}_{\mathfrak{r}}}$ bezüglich durch $\mathfrak{v}_{i;\alpha_1\ldots\alpha_{\mathfrak{r}}}$, $\mathfrak{v}_{i;\bar{\alpha}_1\ldots\bar{\alpha}_{\mathfrak{r}}}$ ersetzt. Ist Π eine Funktion der Grössen $x_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}}$, $\mathfrak{v}_{i;\alpha_1\ldots\alpha_{\mathfrak{q}}}$, so bedeute $\bar{\Pi}$ dieselbe Funktion der $x_{i,\bar{\alpha}_1\ldots\bar{\alpha}_{\mathfrak{r}}}$, $\mathfrak{v}_{i,\bar{\alpha}_1\ldots\bar{\alpha}_{\mathfrak{q}}}$. Π heisst eine *Parameterinvariante* vom Gewichte c, wenn bei allen Transformationen von $\mathfrak{B}$

$$\bar{\Pi} = \mathfrak{D}^c\,\Pi \quad \ldots\ldots\ldots\ldots \quad (35)$$

Veränderliche der Art $\mathfrak{v}_{i;\alpha_1\ldots\alpha_{\mathfrak{r}}}$ sind die Variationen $\delta^{\mathfrak{s}} x_{i,\alpha_1\ldots\alpha_{\mathfrak{r}}} = w_M$ [33]).

---

32) $\mathfrak{r}, \mathfrak{q} = 0, 1, \ldots$.

33) $\mathfrak{r} = 0, 1, \ldots, \rho$; $\mathfrak{s} = 0, 1, \ldots, \sigma$; $M = 1, 2, \ldots$.

da diese bei (5) genau dieselbe Regel befolgen wie die $x_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$ selbst,

$$\delta x_i = \delta \bar{x}_i \;, \quad \delta x_{i,\alpha} = \delta x_{i,\bar{\lambda}} \frac{\partial \bar{u}_\lambda}{\partial u_\alpha}, \ldots, \delta^2 x_i = \delta^2 \bar{x}_i, \ldots \quad (36)$$

$\mathfrak{P}$ sei eine Funktion der w_M; Funktionen dieser Veränderlichen betrifft der

Satz 4. Ist $\mathfrak{P}$ eine Parameterinvariante vom Gewichte c, so gilt dies auch von der Variation $\delta\mathfrak{P}$.

Man beweist ihn entsprechend wie oben den Satz 3, indem man hier von den Abhängigkeiten $x_i = A_i\,(\bar{u}_\alpha, \varepsilon_\mathfrak{a})$ ausgeht. Bei der Variation der Voraussetzung (35) für $\mathfrak{P}$ ist zu berücksichtigen, dass nach (32) $\delta\mathfrak{D} = 0$ ist. In den Formeln, die man jetzt statt der früheren (28^b) erhält, kann wiederum an Stelle der Gesamtheit $\bar{\mathfrak{w}}$ der Werte $\bar{w}_M$ eine *beliebige* $\bar{\mathfrak{w}}$ treten, wenn gleichzeitig $\mathfrak{w}$ durch eine Gesamtheit $\grave{\mathfrak{w}}$ ersetzt wird, die mit $\hat{\bar{\mathfrak{w}}}$ durch (33), (34) zusammenhängt.

Auf Grund der Beziehungen (36) ergibt sich

Satz 5. Ist $\mathfrak{P}$ eine Parameterinvariante vom Gewichte c, so gilt dies auch von der Ableitung $\partial\mathfrak{P}/\partial\delta^{\mathfrak{s}} x_i$.

Beispiel einer Parameterinvariante vom Gewichte 1 ist nach (6) die Funktion G selbst. Wir heben unter den Invarianten Π diejenigen besonderen $\varkappa$ hervor, welche wie G *nur die Veränderlichen* $\mathfrak{d}^{\mathfrak{r}} x$ enthalten. Auf solche Invarianten der Funktion F (14) zielt folgende Ergänzung des Satzes 1 ab:

Satz 6. Wenn unter den Voraussetzungen des Satzes 1 die ξ_i absolut parameterinvariant sind und $h\,(\mathfrak{d}^{\mathfrak{r}} x_i, \xi_i)$ eine Parameterinvariante vom Gewichte c ist, so trifft das letztere auch auf die dort aus h gewonnene Punktinvariante g zu.

Die erwähnte Eigenschaft (6) der Grundfunktion

$$\bar{F} = \mathfrak{D} F \quad \ldots \quad (37)$$

führt nämlich in Verbindung mit der Transformation [1]), (19) der Determinanten θ_i

$$\bar{\theta}_i = \mathfrak{D}\theta_i \quad \ldots \quad (38)$$

zu der Invarianz

$$\frac{\partial \bar{F}}{\partial \bar{\theta}_i} = \frac{\partial F}{\partial \theta_i} \quad \ldots \quad (39)$$

Es können mithin in der für h gültigen Formel (35) die Grössen $\partial F/\partial\theta_i$ an die Stelle der ξ_i treten.

Chemistry. — *Physical purity and powder Röntgenogram.* By N. H. KOLKMEIJER. (Communicated bij Prof. ERNST COHEN).

(Communicated at the meeting of October 29, 1927).

Opinions differ as to the existence of two modifications of mercuric oxide, a red and a yellow one. ERNST COHEN [1]) concludes from the existence of a difference of potential between two electrodes Hg (HgO) KOH, of which HgO with one had the red form, with the other the yellow, that the two forms are allotropic modifications. On the other hand WILH. OSTWALD [2]) and also SCHICK [3]) are of opinion that the difference in colour is the result of a difference in size of the particles, and base this opinion on their experiments on velocity of solution and solubility of the two forms.

LEVI [4]) lately tried to solve this point of difference by taking a powder photo of the two forms with X-rays. From a comparison of the two exposures, he concludes that there can no longer be any doubt as to the crystallographic identity of the two forms of HgO.

Without taking sides in this point of controversy [5]) it appears to me that it is not inexpedient to point out that LEVI's conclusion is not justified by his investigation. When it is not an ascertained fact that the two forms, investigated by LEVI, consist of one modification for 100% — and in his paper not a word is said which shows that he has verified it — it is highly possible that the yellow colour was caused f.i. by the admixture of a certain percentage of the yellow modification with the red. And if this percentage is not high enough, the admixture does not mark its lines on the films in the normal time of exposure (which LEVI, judging from the reproduction of his films has not exceeded).

In order to make this clear we wish to call attention to the fact, that COHEN has more than once pointed out, that when different physical properties are determined, it is not always known whether the preparate consists for 100 % of one modification, and is therefore „physically pure". If the

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1) ERNST COHEN, Z. phys. Chem. **34**, 69, 1900.

2) WILH. OSTWALD, Z. phys. Chem. **17**, 183, 1895; **18**, 159, 1895; **34**, 495, 1900.

3) K. SCHICK, Ibid. **42**, 155, 1903. See also Gay Lussac C.R. **16**, 309, 1843; G. FUSEYA J. Am. Chem. Soc. **42**, 368, 1920; R. VARET C.R. **120**, 622, 1895; Bull. Soc. chim. **[3]**, **13**, 677, 1895; HULETT Z. physik. Chem. **37**, 400, 1901.

4) G. R. LEVI, Gazz. chim. Ital. **54**, 709, 1924. See also R. FRICKE Z. anorg. und allgem. Chem. **166**, 244, 1927.

5) Prof. COHEN told me by word of mouth that he himself inclines to the opinion of OSTWALD and SCHICK.

substance is a labile modification, at the temperature and pressure of the experiment, it is very difficult to get a preparate which does not at least contain a very small percentage of the stable modification. If, on the other hand, the preparate is the stable modification, it is highly possible that it contains a few percents of the labile modification. And in either case the change of labile to stable modification can have a slow course. Each of the two forms of HgO, the red and the yellow, may therefore very well be a mixture of a yellow and a red modification, and if we wish to make reliable determinations of the properties of such substances it is necessary first to make sure of the physical purity. How this can be done by means of the X-rays will appear from what follows.

From the observations made by ANDREWS [1]) in which even a considerable percentage of an alien substance did not appear on the film it is evident that in such a mixture of two modifications of the same substance nothing need appear of the presence of one of them, if the percentage of the latter is not rather great. Even 20 % of Ni in a Ni-Fe alloy did not give lines on the diagram. In order to trace whether similar percentages would also remain invisible if we took two modifications of the same substance, I have, at the request of Prof. COHEN, made exposures of a mixture of 90 % of white with 10 % of gray tin, and of a mixture of 80 % of white with 20 % of gray tin. In volume procents 12 % and 24 % of gray tin were present respectively. It was a happy chance that we could use for it physically pure preparations, which had been prepared by DOUWES DEKKER [2]) for his experiments. With the help of Röntgen photos was demonstrated that the preparations used were indeed physically pure. For even if a preparate contains a small percentage of impurities, only the lines of the principal component are marked off on the film, and precisely as if the preparate had had a purity of 100 %. If we can deduce from the film the structure, we can, at the same time, determine the density of the physically pure modification from measurements made on an impure preparate. If of a preparate, the density is measured with a pycnometer, and if this agrees with the value which was determined in the way mentioned above, we know for certain [3]) that we have a physically pure preparate before us. In the preparates of DOUWES DEKKER the densities of the gray tin, determined with a pycnometer and a Röntgenogram, were 5.765 and 5.764 ; of the white tin 7.285 and 7.29.

In figure 1 we have the prints of our exposures, placed one over the other, of 100 % of gray, 80 % of white with 20 % of gray, 90 % of white with 10 % of gray, and 100 % of white tin [4]). On the one of 20 %

[1]) M. R. ANDREWS, Phys. Rev. (2), 17, 261, 1921.

[2]) K. DOUWES DEKKER, Diss. Utrecht 1927.

[3]) If, at least, the difference in density of the two modifications is not too small. (Cf. A. L. TH. MOESVELD, Chem. Weekbl., 24, 485, 1927).

[4]) During the exposure the temperature was continually about 18°. The temperature of the transitionpoint gray $\rightleftarrows$ white tin is 13°.

of gray tin we see clearly the strongest line of gray tin (indicated with an arrow.) On the print of 10 % of gray tin we can still see a trace of that

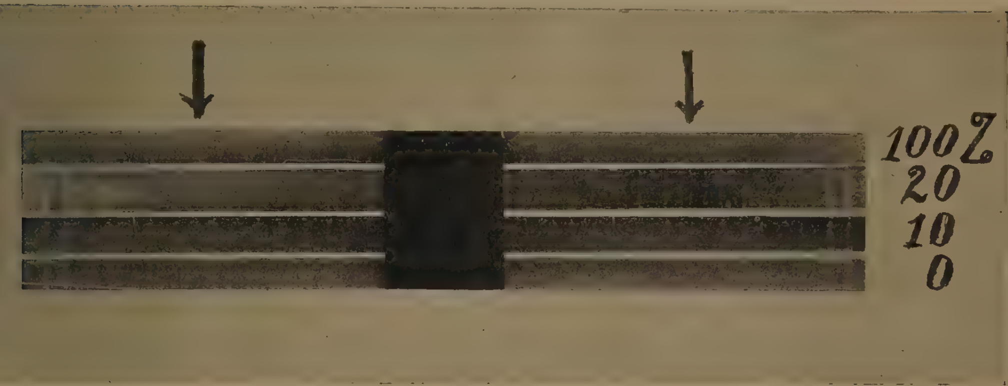

Fig. 1.

line, so indistinct however, that it would, no doubt, have not been observed, if we had not been informed beforehand of the composition of the preparate. This proves that a pollution with as much as 10 % of another modification might have remained unnoticed in LEVI's experiments. Moreover it is very striking that in LEVI's table some rather distinct lines (21 and 41) are given for the yellow substance, which were not found in the red. LEVI does give an explanation for it; but his line of argument does not seem to me very sound. As regards this fact, there is a striking agreement with figure 1 above.

Meanwhile it must be observed that when the time of exposure is ten times longer — apart from the absorption of the rays in the preparate — the lines of the 10 vol. % of admixture ought to be seen as well as the lines of 100 % of admixture with a normal time of exposure. It might happen that, on account of the long exposure the lines of the 90 % of main product would be so much over-exposed, that the lines of the admixture would be rendered invisible. Yet it does not seem impossible to trace sometimes small percentages of a new modification by means of the way indicated. It even seems to be possible to make quantitative determinations with it. Experiments on this subject will be made in this laboratory. We shall also try to make this method serve for quantitative analysis, when the components have a different chemical composition [1]. Qualitatively some results have already been obtained [2].

In cases, in which the densities of the two modifications do not differ

[1]) I.a. L. VEGARD in the case of mixed crystals.

[2]) HULL could distinguish among other things from the photos between the cases that the powder was a mixture of NaCl and KF, and that it was a mixture of NaF and KCl.

very much, so that the other methods for tracing modifications (dilatometer) leave us in the lurch, the Röntgenometric method, given above, may be of great importance. Yet we must not forget that the radiation itself might act as a catalyst, which accelerates the transformation of the labile into the stable modification. This chance, however, is not very great, and perhaps we might also reduce it to a smaller compass by using many greatly different wavelengths.

VAN 'T HOFF-*Laboratory.*

Utrecht, October 1927.

Anatomy. — *The Proportion of cerebellar to total brain weight in Mammals.* By IRMARITA KELLERS PUTNAM [1]) M.D. (Communicated by C. U. ARIËNS KAPPERS).

(Communicated at the meeting of December 17, 1927).

There are few data available upon the proportion between the weight of the cerebellum and that of the whole brain in mammals. Perhaps this is partly because there are few large collections of mammalian brains in existence. Moreover figures which do exist in the literature coming from different sources are not comparable as the cerebella have been removed in no standard fashion and the weighing of the brain has been done under varying conditions and generally with the meninges, at the least with the pia mater and parts of the arachnoid [2]).

Material and Methods.

A hundred mammalian brains of the collection at the Central Institute for Brain Research, Amsterdam, were used for this investigation. The body weights and the ages of the animals were not noted. The collection included old and young animals, but no foetal brains nor brains of new born animals were included. The brains mentioned below had all been preserved, without meninges, for varying lengths of time in 4 % formaldehyde (10 % formaline). Brains in 10 % formaline (4 % formaldehyde), according to FLATAU (1897), the first month of preservation increase from one to two per cent of their original weight.

Between the first and fifth months they loose some of this initial increase, so that at the fifth month they are only one per cent heavier than their original weight and remain so for the following months (FLATAU's observations covered 15 months). Most of the brains in this series were preserved for a still longer period. As however in the last ten months of a fifteen months preservation the then obtained increase of 1 % does not change, there is little reason to believe that it should increase later.

There is a slight percentage difference between weight changes in brain and spinal cord during similar preservation. Whether there is a difference between the change of the cerebellar weight and the rest of the brain we do not know. We can assume however that this is very slight.

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[1]) Holder of the Vassar Alumnae Fellowship for graduate study, 1925—1926.

[2]) Weighing the brain with the pia (and part of the arachnoid) includes a source of errors, as this tissue may keep a fairly great deal of the preservation fluid.

In alcohol on the other hand brains loose large amounts of weight and the loss is continuous, so that at the end of fifteen months a brain has lost thirty four per cent of its original weight. We have listed only one brain preserved in this way (cf. p. 162).

The hypophysis, which was often allowed to remain in the sella turcica, when the brain was removed from the skull in the zoological garden, was dissected off those brains where it was still attached in order to equalize conditions. *All meningeal tissue was carefully removed* [1]).

The cerebellar peduncles were severed just above the emergence of the seventh and eigth nerves tangential to the brain stem, care being observed to leave the posterior corpora quadrigemina and the fourth nerve intact. The brain stem was cut 3 mm below the calamus scriptorius. Pains were taken to perform these manipulations as nearly as possible in the same way, as it is obvious that each one of them involves a source of error.

Of some of the brains only one half was available the other half being cut in microscopical sections. In these cases the *hemisection* is mentioned in my tables. They were only used for this statistic if it appeared that the hemisection was accurately made, and thus did not influence the percentage relation.

Weighing was done on a chemical balance sensitive to a milligram. The brains were removed from formaldehyde, dried with a soft towel (so that there was no more draining of fluid) and weighed directly, exposed to the air during the process. Very little time was required for this, since preliminary weighings had been made on the previous day to facilitate the final weighing.

It was found that one individual could so standardize the amount of drying and that there was less variation between two weighings of the same brain on successive days by this method than by either of the methods described in the next paragraph. Also it was found that the variations for small brains were less than for large brains which was the reverse with the other two methods. This is important because very slight changes in the weights of the small brains cause large percentage variations, while the reverse is the case for the large brains. However, these variations also constitute a source of error.

Dr. RICHARD S. LYMAN of Rochester University, New-York to whom I am indebted for much help, determined during his sojourn in the Central Institute for Brain Research the rate of loss of moisture when brains were allowed to dry in the open air, and found that a constant state of dryness was not reached at any point. As was to be expected the rate of loss was proportional to the surface area so that the cerebellum lost weight more rapidly than the rest of the brain and small brains more rapidly than large ones. An attempt was also made to bring the brains to a constant state of moisture by keeping them in a moist chamber.

While the daily variations of these brains were less than the hourly variations which Dr. LYMAN obtained, they were still greater than the variations obtained by the method

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[1]) This may explain that the brain weight of several animals is less than the figures mentioned in the current litterature.

described in the paragraph above, which was therefore chosen for the series. Comparative results of the three methods applied to a brain of a young *Nasua narica* (preserved in formaline 10 %) are recorded below to establish the justification of this choice. (Table I).

I. Brain exposed to air during five hours (Determinations by R. S. Lyman).

Moment of weighing	Cerebellar w.	Cerebral w.	Total brain w.	Cerebel. perc.
Aug. 13th 11.35 AM.	2.83 gr.	20.73 gr.	23.56 gr.	12.00 %
„ „ 12.00 M.	2.79 „	20.59 „	23.38 „	11.93 „
„ „ 12.30 PM.	2.75 „	20.46 „	23.21 „	11.84 „
„ „ 1.00 „	2.73 „	20.37 „	23.10 „	11.82 „
„ „ 1.30 „	2.70 „	20.20 „	22.96 „	11.76 „
„ „ 2.30 „	2.65 „	20.09 „	22.74 „	11.64 „
„ „ 3.30 „	2.60 „	19.93 „	22.53 „	11.53 „
„ „ 4.30 „	2.55 „	19.77 „	22.32 „	11.42 „
II. Brain kept in moist chamber.				
Sept. 5th	2.78 gr.	20.22 gr.	23.00 gr.	12.20 %
„ 6th 11.00 AM.	2.745 „	20.155 „	22.90 „	12.0 „
„ „ 4.00 PM.	2.74 „	20.04 „	22.78 „	12.05 „
„ 7th	2.72 „	20.00 „	22.72 „	11.99 „
„ 12th	2.67 „	19.72 „	22.39 „	11.96 „
„ 14th	2.64 „	19.58 „	22.22 „	11.88 „
III. Brain removed from formaldehyde softly dried and weighed directly				
Sept. 15th	2.73 gr.	20.53 gr.	23.26 gr.	11.73 %
„ 16th	2.74 „	20.57 „	23.31 „	11.70 „
„ 17th	2.73 „	20.54 „	23.27 „	11.73 „

The method of weighing under water might be still more accurate. However, this method was not tried as it was thought too elaborate for this purpose. The temperature of the water influencing the specific gravity of the particular water used, movements in the water must all be carefully controlled, if this method is to be as accurate practically for our purpose as it is theoretically.

The results of the weighings are given in the accompanying table, which shows the weight of the cerebellum, that of the cerebral hemispheres and stem, the total brain weight and the proportion between the weight of the cerebellum and that of the total brain, expressed as a percentage of the latter.

The classification of the animals is essentially that of OSBORN (1910). The primates however, are listed according ELLIOT (1913).

ORDER RODENTIA.

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Sciuridae.				
Pteromys nitidus	1.39 gr.	7.50 gr.	8.89 gr.	15.62 %
Cynomys ludovicianus	1.5 „	9.95 „	11.45 „	13.10 „
Echinosciurus aureogaster	.96 „	5.24 „	6.20 „	15.45 „
Heterosciurus notatus	.65 „	3.63 „	4.28 „	15.18 „
Leporidae.				
Lepus cuniculus	.94 „	6.07 „	7.03 „	13.4 „
Hystricidae.				
Hystrix cristata	2.9 „	16.3 „	19.2 „	15.00 „
Hystrix javanica (hemisect.)	1.39 „	8.23 „	9.62 „	14.45 „
Coendu prehensilis	2.57 „	16.26 „	18.83 „	13.62 „
Chinchillidae.				
Lagostomus trichodactylus	1.9 „	13.48 „	15.38 „	12.35 „
Dasyproctidae.				
Dasyprocta aguti	2.32 „	17.58 „	19.90 „	11.65 „
Dasyprocta aguti	2.08 „	13.73 „	15.81 „	13.12 „
Octodontidae.				
Myopotamus coïpu	1.13 „	10.33 „	11.46 „	9.80 „

ORDER EDENTATA.

Myrmecophagidae.				
Myrmecophaga jubata	8.67 gr.	43.4 gr.	52.07 gr.	16.52 %
Bradipodidae.				
Choloepus didactylus	5.27 „	25.68 „	30.95 „	13.84 „
Choloepus didactylus	4.5 „	24.31 „	28.81 „	15.55 „
Choloepus didactylus (hemisection)	2.3 „	12.16 „	14.46 „	15.90 „
Dasypodidae.				
Dasypus villosus	2.19 „	12.86 „	15.05 „	14.53 „

ORDER UNGULATA.

Sub Order Artiodactyla.	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Dicotylidae.				
Dicotyles labiatus	7.67 gr.	58.98 gr.	66.65 gr.	11.55 %
Camelidae.				
Camelus dromedarius	57.5 „	420.— „	477.5 „	12.05 „
Auchenia glama	20.23 „	128.85 „	149.08 „	13.60 „
Giraffidae.				
Camelopardalis giraffa (young)	53.3 „	433.— „	486.3 „	10.90 „
Cervidae.				
Cariacus nemoralis	13 7 „	114.4 „	128.1 „	10.70 „
Alces machlis	27.8 „	232.5 „	260.3 „	10.65 „
Cervulus muntjac (young spec.)	4.45 „	39.5 „	43.95 „	10.12 „
Rusa hippelaphus	19.37 „	116.15 „	185.52 „	10.4 „
Rusa hippelaphus	19.7 „	166.2 „	185.52 „	10.6 „
Rucervus Eldi	21.9 „	179.7 „	201.6 „	10.86 „
Dama dama	14.9 „	146.5 „	161.4 „	9.22 „
Ovidae.				
Ovis tragelaphus (small spec., hemisect.)	12.1 „	107.15 „	119.25 „	10.1 „
Ovis tragelaphus	19 „	173.— „	192.— „	9.87 „
Capra hircus (small spec.)	10.35 „	73.6 „	83.95 „	12.32 „
Antilopes.				
Antilope cervicapra (hemisect.)	5.35 „	44.5 „	49.85 „	10.75 „
Antilope borea	6.3 „	53.95 „	60.25 „	10.45 „
Oreas Livingstoni	37.2 „	395.4 „	432.6 „	8.65 „
Catoblepas gnu (small spec., hemis.)	7.26 „	66.3 „	73.56 „	9.86 „
Anoa depressicornis	18.7 „	163.3 „	182.— „	10.26 „

Sub Order Perissodactyla.

Equidae.				
Equus caballus	57 gr.	446 gr.	503 gr.	11.3 %
Equus asinus	36.5 „	298.2 „	334.7 „	10.92 „
Tapiridae.				
Tapirus indicus	27.77 „	213.12 „	240.89 „	13.00 „

ORDER PROBOSCIDAE.

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Elephas indicus (small specim.)	923.— gr.	2816.—gr.	3739.—gr.	24.68 %
ORDER ODONCETI.				
Delphinidae.				
Phocaena phocaena (hemisection)	28.85 gr.	157.86 gr.	186.71 gr.	15.49 %
Phocaena phocaena	58.— „	332.— „	390.— „	15.— „
ORDER CARNIVORA.				
Ursidae.				
Ursus arctos (young sp.)	36.— gr.	196.— gr.	232.— gr.	16.30 %
Ursus maritimus	68.8 „	365.5 „	434.3 „	15.75 „
Heliarctos malayanus	40.7 „	211.65 „	252.35 „	16.10 „
Mustelidae.				
Lutra vulgaris (hemisection)	2.05 „	22.— „	24.05 „	8.38 „
Lutra vulgaris (hemisection)	1.94 „	16.3 „	18.24 „	10.69 „
Putorius putorius	0.61 „	4.77 „	5.38 „	11.30 „
Mustela erminea (hemisection)	0.3 „	2.27 „	2.57 „	11.66 „
Mustela erminea (hemisection)	0.22 „	1.84 „	2.06 „	10.65 „
Mustela foina (hemisection)	0.40 „	2.81 „	3.21 „	12.60 „
Meles taxus	5.39 „	38.42 „	43.81 „	12.29 „
Viverridae.				
Paradoxurus musanga (hemisection)	1.22 „	7.35 „	8.57 „	14.28 „
Arctitis binturong (hemisection)	2.17 „	12.63 „	14.80 „	14.65 „
Herpestes griseus	1.18 „	9.38 „	10.56 „	11.17 „
Canidae.				
Canis familiaris	4.55 „	46.75 „	51.30 „	8.9 „
Canis familiaris	7.— „	66.93 „	73.93 „	9.6 „
Black and Tan (hemisection)	3.82 „	33.70 „	37.50 „	10.18 „
Retriever (hemisection)	4.24 „	42.12 „	46.36 „	9.15 „
Dachshund	5.62 „	62.81 „	68.43 „	8.25 „
Spaniel	8.57 „	81.7 „	90.27 „	9.50 „
Airdale Terrier	7.18 „	72.65 „	79.83 „	9.17 „

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Wolf Hound (hemisection)	3.81 gr.	35.97 gr.	39.78 gr.	9.60 %
Shepherd Dog (hemisection)	3.18 „	32.7 „	35.88 „	8.85 „
Gordon Setter (hemisection)	3.80 „	41.80 „	45.60 „	8.46 „
Collie (hemisection)	4.2 „	35.66 „	39.26 „	10.62 „
German Dog	10.2 „	87.42 „	97.62 „	10.4 „
Irish Setter	7.94 „	73.62 „	81.56 „	9.70 „
Boxer (hemisection)	3.28 „	32.72 „	36.— „	9.02 „
Canis lupus	5.67 „	58.43 „	64.10 „	8.85 „
Vulpus lupus opus (hemisection)	1.64 „	12.11 „	13.75 „	11.90 „
Felidae.				
Zibethailurus pardalis (hemisection)	2.57 „	22.39 „	24.96 „	10.30 „
Zibethailurus pardalis	6.98 „	44.51 „	51.49 „	13.56 „
Felis leo	19.65 „	167.24 „	186.92 „	10.52 „
Felis leo (hemisection)	8.78 „	86.95 „	95.73 „	9.28 „
Felis concolor (hemisection)	7.53 „	48.11 „	55.64 „	13.05 „
Felis concolor (hemisection)	5.78 „	45.11 „	50.89 „	11.36 „
Felis macrosceles nebulosa	6.13 „	39.03 „	45.16 „	13.52 „
Lynx lynx	5.78 „	36.31 „	42.09 „	13.70 „
Phocidae.				
Phoca vitulina (small spec.)	26.10 „	143.40 „	169.50 „	15.4 „
Phoca vitulina	29.5 „	179.— „	208.50 „	14.1 „

Order Primates.

Prosimiae.				
Daubentonidae.				
Cheiromys madagascariensis (sm. spec.)	3.38 gr.	22.23 gr.	25.61 gr.	13.18 %
Lemuridae.				
Lemur catta	1.38 „	9.03 „	10.41 „	13.35 „
Lemur macaco (hemisection)	1.42 „	9.— „	10.42 „	13.60 „
Lemur mongoz (hemisection)	1.40 „	9.17 „	10.57 „	13.23 „

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Simiae.				
Callitrichidae.				
Callithrix pygmaea	4.66 gr.	37.61 gr.	42.27 gr.	11.— %
Hapale jacchus	.25 „	2.6 „	2.85 „	8.75 „
Oedipomidas oedipus, Oedipomidas oedipus (average of two hemisections)	.8 „	6.85 „	7.65 „	10.375 „
Cebidae.				
Mycetes laniger	.55 „	6.03 „	6.58 „	8.36 „
Chrysothrix sciureus	3.76 „	28.30 „	32.06 „	11.70 „
Nyctipithecus trivirgatus	.38 „	3.06 „	3.44 „	11.06 „
Ateles ater (hemisect.)	4.15 „	34.— „	38.15 „	10.86 „
Lagothrix lagotricha	9.7 „	76.58 „	86.28 „	11.22 „
Cebus hypoleucus (hemisect.)	2.32 „	23.2 „	25.52 „	9.09 } 10.88
Cebus hypoleucus (hemisect.)	2.6 „	17.92 „	20.52 „	12.68 } 10.88
Cebus fatuellus, Cebus fatuellus (average of two hemisections)	2.78 „	25.10 „	27.88 „	9.95 „
Lasiopygidae.				
Cynocephalus hamadryas	10.15 „	84.50 „	94.65 „	10.70 „
Cynocephalus porcarius (hemisection)	7.8 „	67.88 „	75.68 „	10.30 „
Macacus rhesus	7.04 „	70.6 „	77.64 „	9.05 „
Macacus rhesus	6.07 „	65.3 „	71.37 „	8.65 „
Cercocebus fuliginosus	1.94 „	16.3 „	18.24 „	10.62 „
Mona mona (hemisection)	2.39 „	24.8 „	27.19 „	8.80 „
Inuus inuus	7.03 „	72.53 „	79.56 „	8.80 „
Cercopithecus callitrichus	5.42 „	50.26 „	55.68 „	9.70 „
Erythrocebus patas	6.93 „	64.76 „	71.69 „	9.66 „
Semnopithecus entellus (hemisection)	4.17 „	38.61 „	42.78 „	9.75 „
Anthropoidae.				
Simia satyrus	30.8 „	191.4 „	222.2 „	13.86 „
Simia satyrus (hemisection)	14.11 „	92.32 „	106.43 „	13.25 „
Troglodytes niger (hemisect.)	18.1 „	102.5 „	120.6 „	15.— } 14.05
Troglodytes niger (hemisect, alcohol)	17.27 „	114.19 „	131.46 „	13.11 } 14.05

Discussion.

The establishment of the percentage variations in individuals is a necessary prelimenary to any interpretation of figures involving the larger groups. Unfortunately this collection contains few duplicates. Where more than one individual of a species is present, the maximum difference between the percentage weights is 3.59 as may be seen in the following table.

III. Percentage variation between different individuals of the same species.

Species	Individual	Percentage	Differences in percentage
Ovis tragelaphus	I	10.10	0.23
	II	9.87	
Macacus rhesus	I	9.05	0.40
	II	8.65	
Simia satyrus	I	13.86	0.61
	II	13.25	
Mustela erminea	I	11.66	1.01
	II	10.65	
Felis leo	I	10.52	1.24
	II	9.28	
Phoca vitulina	I	15.4	1.30
	II	14.1	
Dasyprocta aguti	I	13.12	1.47
	II	11.65	
Felis concolor	I	13.05	1.69
	II	11.36	
Rusa hippelaphus	I	10.60	0.20
	II	10.40	
Troglodytes niger	I	15.—	1.89
	II	13.11	
Choloepus didactylus	I	15.90	2.06
	II	15.55	
	III	13.84	
Lutra vulgaris	I	10.69	2.31
	II	8.38	
Zibethailurus pardalis	I	13.56	3.26
	II	10.30	
Cebus hypoleucus	I	12.68	3.49
	II	9.08	

In connection with this percentage variation among individuals, it is profitable to consider the figures reported by ARIËNS KAPPERS (1926), who also found a considerable range of variation amongst man.

The percentages of cerebellar to total brainweight in the brains

of 25 Dutchmen were	8 %	to 12.6 %	a range of 4.6 ;
of 22 Chinese	8.61 %	to 12.22 %	a range of 3.61 ;
of 8 Japanese	9.51 %	to 11.25 %	a range of 1.74.

It is very difficult to tell the cause of these variations and highly improbable that the cause in each case is the same.

For man WEISBACH (1867), who made a similar observation, believed the heavier specimens to have the greater cerebellar percentage.

The question wether body size has any influence on the cerebellar percentage of animals can be best controlled by comparing smaller and larger, though both adult representatives of the same species as enumerated in table III.

Of the animals mentioned there only of the two Mustela erminea, the two Zibethailurus pardalis and the two Felis concolor and two Simia satyrus, the largest specimens (according to the total brain weight) had larger cerebella.

On the other hand, however, of the two Dasyproctae, three Choloepus, two Ovis tragelaphus, two Lutrae, two Phocae and two Phocaenae, the specimen with the greatest (total brain) weight had a smaller percentage of cerebellum. From this no evidence can be obtained in favor of a constant influence of the bodysize (or total brainweight) in the percentage of the cerebellum.

Also KAPPERS could not confirm WEISBACH's opinion — that a larger weight should be constantly correlated with a larger cerebellum, although this occassionally occurs.

I have also made a comparison between the different representatives of the same order, suborder or genus wherever more than two specimens were available, just as I did in the cases of species. The advantage of this comparison is moreover that the differences in size are greater and more constant though certainly also other factors come in here (vide infra).

Here also it is evident that WEISBACH's thesis does not hold good as in the majority of cases the smaller genus has a higher percentage. So in the rodent suborder of Sciuridae the largest of all, Cynomus ludovicianus, has a cerebellar percentage of 13.10 %, whereas the average of the smaller Pteromys, Heterosciurus and Echinosciurus is more than 15.40 %. Amongst Antilopes [1]) the large Oreas Livingstoni has only 8.65 %, while all the others have about 10 % or more, the small Antilope cervicapra even 10.75 %, the highest percentage amongst this suborder.

The same is observed comparing the Camel (12.05 %) with the smaller Lama (13.60 %). The Giraffa, still larger than the Camel, has only 10.90 %.

Amongst the Ovidae the smaller Capra hircus has a higher percentage than the larger Ovis tragelaphus. In the suborder of the Perissodactyla the smaller Tapir has a higher percentage than Equus caballus and asinus.

[1]) Amongst the Cervidae the percentage varies so little that this suborder seems to be less fit for comparison. Its result would not be in favor of any rule in this respect.

Amongst the carnivorous suborder of the Ursidae we find that the largest representative Ursus maritimus has less cerebellum than both others, although in this whole suborder the percentage is very high (vide infra).

In the Mustelidae[1]) and Canidae no constant rule can be observed. It is however striking that of Canis lupus, Vulpes lupus and Vulpus lupus opus the latter, the smallest, again has the highest percentage. The same holds good if we compare the average figure of the two Felis leo, with the average figure of the two Felis pardalis and the average figure of the two Felis concolor, the average cerebellar percentage of the lion, which is the greatest animal of this family, being the smallest.

Amongst Prosimiae the smaller Lemurs have a slightly higher percentage than Chiromys.

So we see that if there is any rule, it is certainly not in favor of WEISBACH's conception but more likely in favor of the higher percentage in the smaller representants.

That, however, also this is not constant appears amongst others from the figure of Myrmecophaga compared to Choloepus and the figure of Cercocebus fuliginosus compared to those of Cynocephalus hamadryas and Anthropoids. From this results that other factors, than the size of the body excercize a considerable influence on this figure.

Among these factors are the different cephalization coefficient of different animals and some physiological differences that cannot be expressed in matter of cephalization.

Considering the percentages of cerebellar weight, we have to realize that this percentage may change as well by variation in the forebrain development as by variation in the weight of the cerebellum itself.

Variation in forebrain development will chiefly occur between orders and suborders where the cephalization index is very different, as this cephalization index largely depends on the forebrain, since this represents the greater mass of the encephalon.

So it may be explained that the average cerebellar % in man is only 10.5 % (KAPPERS) while in Anthropoids it is 13.72 %.

The question however arises if greater cephalization *necessarily* diminishes the cerebellar percentage. If for this we consider the different orders it appears that although in each order there are considerable variations (see below) some of the *highly* cephalized animals are conspicuous by their *large* cerebellar percentage.

Among these are the Proboscidae, Pinnipedii, Odontoceti and Edentata, compared to their next relatives. In the Proboscidae, Pinnipedii and Odontoceti this fact is the more striking as their cephalization also is very considerable (according to DUBOIS) compared to their next relatives.

We may conclude from this that in these animals the motor synergia has acquired an extraordinary precision. So in the Elephant the ability for complicated movements may be related to the large cerebellar per-

[1]) For the Viverridae see below.

centage. This animal possesses very precise independent monolateral movements of its extremities and a very fine adjustment of its trumpet.

In the Pinnepedii and Cetacea it is chiefly the swimming movement that involves a great cerebellar capacity. The agility of sealions is well known as also their equilibric acrobatics outside the water, often shown to the public. In Dolphins, who easily swim around a quickly moving steamer, the motile capacities are equally striking.

Still in both animals the cerebellar organization is very different. In the Pinnipedii, who greatly use their forelegs, the hemispheres of the cerebellum (the center of independent movements of the legs, BOLK) are increased. In the Cetacea, who have no extremities and where the strong tail is the sole moving agent, the pars floccularis is enlarged and the paraflocculus is enormous (BOLK) on account of its pontine connections (R. B. WILSON) and is chiefly responsable for the great size of the cerebellum. Now it is striking that in the whale, Balaenoptera sulfurea KAPPERS found a still higher percentage of cerebellum (18.95 %) than I did in Phocaena (15 %). Still the motile capacities of Balaenoptera are not nearly the same as those of Phocaena, as they hardly can follow a fast steamer. So I am inclined to believe that the higher cerebellar index in Balaenoptera is influenced by its cephalization index (which according to DUBOIS is only $^2/_3$ of that of the Odontoceti), as it is very probable that this smaller cephalization is largely due to the comparatively smaller forebrain in Balaeonoptera, which is also more dolichocephalous than the forebrain of Odontoceti, whose greater brachencephaly, according to KAPPERS (1927), is also a result of greater forebrain development.

As far as concerns the high cerebellar index in Edentates this fact may be due to the special character of their movements, which though being extremely slow, are highly complicated and require much independency of each extremity.

The accuracy of movements, even if slow, is of importance here as this involves a great deal of inhibition and synergia, which, as we know, are located in the cerebellum (TILNEY and RILEY). That Myrmecophaga has a still higher percentage than the other Edentates examined may be explaned by the fact that in addition to its extremities it has a very long snout, which is used as a sort of trumpet, for gathering food, as does the Elephant, and certainly has great proprioceptive capacities.

The influence of the character of the motitility of an animal is also seen in the Cervidae, which though being quick, have a great simplicity of gait, their extremities acting in a rather monotonous rythmic collaboration (alternation) of both sides. They have the lowest average of cerebellar percentage.

Among the order of Carnivora, there is a striking and fairly constant difference between the various suborders, the cerebellar percentage being the largest in the Ursidae, next come the Viverridae, then the Felidae, followed by the Mustelidae and finally the Canidae.

It is interesting however, that of the Carnivora the Ursidae also have the highest cephalization index, which might make us expect (cf. p. 165) that their cerebellar index should be smaller, as in the comparison of Odontoceti and Balaenoptera the cerebellar percentage is smaller in Odontoceti.

It is however well known that the capacities in finer adjustment of independent movements of the limbs and in conformity the proprioceptive instrument are very highly developed in bears. As they also have the greatest cephalization (according to DUBOIS Ursus malayanus holds a position amongst Carnivora as the anthropoids do amongst monkeys), we have here a case similar to that of the Elephant. As in the latter it is not improbable that in the cephalization of Ursus these proprioceptive functions, which are projected both on the cerebrum and on the cerebellum, but which are preponderant for the cerebellum, act the largest part.

On the other hand the relatively high cerebellar percentage of Viverridae is more likely due to the smaller development of their forebrain in comparison to other carnivora, the Viverridae having the lowest cephalization among Carnivora (DUBOIS).

This would also explain their relation to the Mustelidae, which as a rule are more cephalized [1]).

A very interesting phenomenon is offered by comparing Canidae and Felidae, two genus in which the coefficient of cephalization is practically the same.

Still we see that the Felidae have a higher average figure for the cerebellar percentage as dogs have. In none of the dogs the % mounts higher than 10¾ % whilst in the Felidae the average is far above this figure, rising even to 13.70 %.

This difference no doubt should be explained by the difference in motile abilities in both genus, those of the Felidae being doubtless much more developed than in dogs, specially as far as concerns finer adjustment of independent (unilateral) movements of each of the forelegs.

From the view point attained in this report it is an interesting fact, that, just as the Elephants and the sealions (Pinnipedii), also the Ursidae and Felidae belong to those animals to whom tricks of equilibration can be best taught.

In the *Primates* the Prosimiae have a greater cerebellar % than the real monkeys. As their forebrain (which does not entirely cover the cerebellum, as it does in most monkeys) is relatively small, this might explain the higher cerebellar %, rather than a difference in motile abilities, that are great in both.

On the other hand the highest cerebellar percentage amongst all Primates is found in the Anthropoids, who at the same time are more cephalized, so that as in the Elephant, Ursidae, Felidae and Pinnipedii, this high cerebellar

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[1]) The smaller % in the latter does not necessarily include a smaller development of the cerebellum in relation to the body.

percentage can only be explained by their special cerebellar proprioceptive capacities.

From what is said above it appears that just as the Ursidae, Felidae and the Elephant and Odontoceti, so the Anthropoids, more than other monkeys should be considered as special cerebellar animals, — since in spite of their higher cephalization their cerebellar percentage is greater than that of their next relatives. On the other hand the fact that men have a cerebellar index smaller than Anthropoids should be ascribed by the greater development of the forebrain.

Summary.

1°. A comparison of the proportion between the weight of the cerebellum and the total brainweight in a series of mammals shows no constant correlation between the size of the animal and the proportionate weight of the cerebellum.

2°. Factors such as cephalization coefficient and capacities of adjustment of the extremeties, tail or trumpet have the greatest influence in the relative proportions.

3°. Animals. naturally endowed with *special* motile capacities, including those used for special motile tricks such as the elephant, sealions, cats, bears and anthropoids have the greatest cerebellar percentage.

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Chemistry. — *Osmosis of ternary liquids. Experimental part,* II. By F. A. H. SCHREINEMAKERS and B. C. VAN BALEN WALTER.

(Communicated at the meeting of February 25, 1928).

In the preceeding [1]) communication (Exp. I) we have discussed the apparent osmosis of the systems I—V consisting of $NaCl + Na_2\,CO_3 + H_2O$; now we shall discuss the apparent osmosis of systems, consisting of $Na_2S_2O_6 + BaS_2O_6 + H_2O$; we shall call these the systems VI—XII.

The left side liquid L_1 of these systems

$$L_1 \mid L'_1 \quad \ldots \ldots \ldots \ldots \quad (1)$$

only contains $H_2O + BaS_2O_6$ (dithionate of barium), the right side liquid L'_1 only $H_2O + Na_2S_2O_6$ (dithionate of natrium). If in figs. 1 and 2 we draw the amount of $Na_2S_2O_6$ on the X-axis and on the Y-axis the amount of BaS_2O_6 of these liquids, then L_1 is represented, therefore, by a point 1 on the Y-axis and L'_1 by a point $1'$ on the X-axis.

It appears from the tables VI and VII that the paths VI and VII (fig. 1) have approximately the same points 1 and $1'$; the same obtains

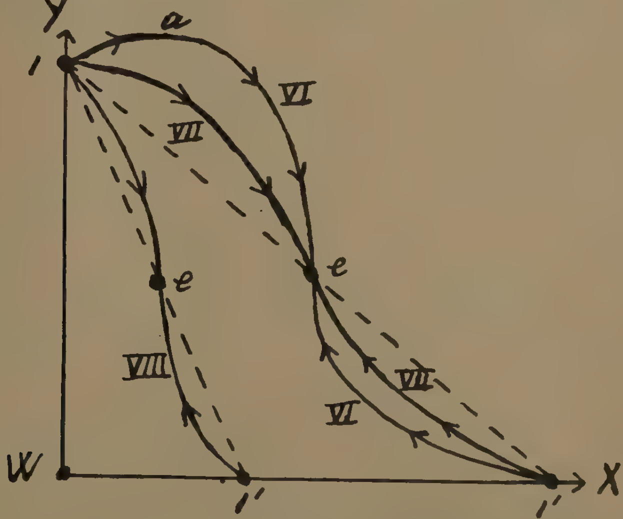

Fig. 1.

for the paths VIII—X, of which only path VIII has been drawn in

[1]) The communications with the "General consideration" are cited as: Gen. I, Gen. II etc.; those with the "Experimental part" as: Exp. I etc.

fig. 1; their points 1 also coincide approximately with those of the other paths in fig. 1; the points 1 and 1′ of the paths XI and XII in fig. 2 also coincide approximately.

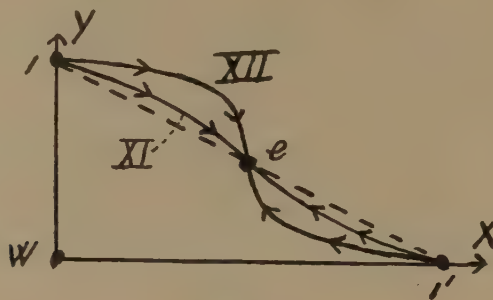

Fig. 2.

We shall first consider the X ($Na_2S_2O_6$)-amount of these systems. It appears from the tables and also from the form of the paths in fig. 1 and 2 that during the whole osmosis in each system the X-amount is smaller on the left side of the membrane than on the right side, that it increases continuously on the left and decreases continuously on the right. For all these systems VI—XII, therefore, obtains the symbol:

$$\uparrow < \downarrow \quad \ldots\ldots\ldots\ldots \quad (2)$$

Consequently during the whole osmosis the X-amount changes normally—normally.

So, the $Na_2S_2O_6$ in these systems behaves in the same way as the $NaCl$ in the systems I—V (Exp. I). This also appears when we draw the $X \, . \, t$-diagrams of these systems VI—XII with the aid of the tables; then we see that they can be represented by fig. 2 (Gen. II), which also schematically represents the Xt-diagrams of the systems I—V.

We now consider the Y (BaS_2O_6)-amount of the systems VII—XII; that of system VI we shall leave out of consideration for the present. From the tables and also from the figs. 1 and 2 it appears that the change can be represented by:

$$\downarrow > \uparrow \quad \ldots\ldots\ldots\ldots \quad (3)$$

The Y-amount of the systems VII—XII changes, therefore, normally—normally during the whole osmosis.

If we draw the Yt-diagrams of these systems with the aid of the tables, then we see that they can be represented by fig. 2 (Exp. I), which also schematically represent the $Y \, . \, t$-diagrams of the systems I—V.

Otherwise it is, however, with the Y-amount of system VI. It appears namely from table VI that this amount increases in the left side liquid,

starting from 1, becomes a maximum in the determination noted with a +, and decreases afterwards; the Y-amount of the right side liquid, however, continuously increases. Consequently branch $1 . e$ of path VI consists of an ascending part $1 . a$ and a descending part a.e.; in point a the Y-amount is a maximum.

For part $1 . a$ (and $1' . a'$) of the path obtains, therefore:

$$\uparrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and for the further part $a . e$ (and $a' . e$):

$$\downarrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So we find: the $Y (BaS_2O_6)$-amount of system VI changes:

on part $1 . a$: anormally—normally
„ „ $a . e$: normally—normally

In point a between the symbols (4) and (5) occurs the transition:

$$| > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This expresses that during an infinitely small time dt the Y-amount of the left side liquid remains constant, while that of the right side liquid increases.

In scheme VI this transition-form has not been given neither in any of the other schemes, it appears from the table that it must be situated on part $3 . 4$ of the path.

If, with the aid of table VI we draw the $Y . t$-diagram of this system, then we see that this can be represented schematically by fig. 3 (Gen. II); (of course the letter m and the ciphers 2, 3, etc. of this figure do not relate to the determinations in table VI).

In the systems I—V (Exp. I) and, as we shall see further, also in the systems VI, VII and XII, anormal changes of the W-amount occur; in all these systems I—XII, however, we only find one single example of anormal change af the concentration of an other substance; this is the change of the BaS_2O_6-amount in system VI. Later on we shall discuss systems, in which the concentration of $NaCl$, NH_4Cl and other substances changes anormally,

In order to discuss the W-amount of the systems, we divide them into three groups; first we take group VIII—X, of which only path VIII has been drawn.

If we deduce the W-amount of the liquids from the tables VIII—X, then it becomes apparent that: during the whole osmosis the W-amount on the left side of the membrane is smaller than on the right side; it increases continuously on the left side of the membrane and it decreases on the right.

This also appears from path VIII in fig. 1. If we draw an imaginary line through e parallel to the side XY, then we see that branch $1 . e$

runs entirely above this line and branch $1'.e$ entirely below this line; we also see then that liquid 1 has a smaller W-amount and $1'a$ a larger W-amount than the final-liquid e; further it appears from the form of the path that there are no tangents, parallel to the side XY, so that neither a minimum-nor a maximum W-amount occurs.

So for these systems obtains what we also find given in the schemes VIII—X sub W viz. the symbol:

$$\uparrow < \downarrow \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Consequently during the whole osmosis the W-amount of the systems VIII—X changes normally—normally.

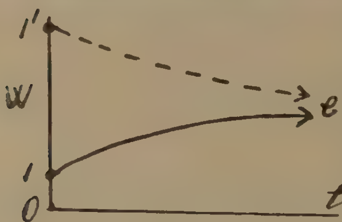

Fig. 3.

If we draw the W. t-diagrams of these systems, we see that they can be represented schematically by fig. 3. Consequently we have already found three different types of Wt-diagrams, viz. fig. 4 (Gen. II), fig. 3 (Exp. I) and fig. 3 in this communication.

We now take system XI; it appears from the path in fig. 2 or from table XI that for the change of the W-amount obtains the symbol:

$$\downarrow > \uparrow \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

So the W-amount of this system changes during the whole osmosis normally—normally.

We may also represent the Wt-diagram of this system schematically by fig. 3; then, however, we must interchange the points 1 and 1', dott the fully-drawn curve and draw the dotted curve in full.

It appears from figs. 1 and 2 that the paths VI, VII and XII have forms corresponding with path V in fig. 1 (Exp. I) and the paths II—IV which have not been drawn here.

If namely through point e a line is drawn parallel to the side XY, this will intersect the path in two points; these points represent the two liquids q and q' with the same W-amount. We can also draw a tangent, parallel to XY, in a point m of branch $1.e$ and in a point M' of branch $1'.e$; consequently the W-amount of the left side liquid is a minimum in m and that of the right side liquid is a maximum in M'.

If from the tables we deduce the W-amount of the liquids, then we are able to find the position of these points q, m and M'.

We now find that the W-change during the whole osmosis can be represented successively by the symbols:

$$\downarrow > \uparrow \qquad {}^{*}\!\downarrow < \uparrow^{*} \qquad \uparrow < \downarrow \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

We find them indicated sub W in the schemes VI, VII and XII. So we find:

the W-amount of the systems VI, VII and XII changes

a. in the beginning of the osmosis: normally—normally;
b. afterwards: anormally—anormally;
c. at last: normally—normally.

Between the first and the second symbol we have the transition;

$$*\!\downarrow\;=\;\uparrow\!* \qquad \qquad (10)$$

this occurs in the points q and q'. Between the second and the third symbol one of the transitions (6) or (7) (Exp. I) occurs. These transitions have not been given in the schemes.

Again the $W.t$-diagrams of these systems can be represented schematically by fig. 4 (Gen. II).

It appears from schemes I—XII:

the X-amount of all systems changes normally;

the Y-amount changes normally in all systems, except in VI where it also changes anormally;

the W-amount changes normally only in the systems VIII—XI, in the others it also changes anormally.

The paths XI and XII (fig. 2) have approximately the same points 1 and 1′. In XI a membrane of collodion, was used, in XII the membrane consisted of collodion in which a deposit of $Cu_2Fe(CN)_6$. Fig. 2 shows very clearly that both systems consequently travel along very different paths; we also see from the schemes that the W-amount in XI changes only normally, whereas in XII this occurs also anormally.

It also appears from the tables that the liquids in XI change their compositions much more quickly than in XII; we see e.g. that the X-amount of the left side liquid

of XI increases in 99 hours already with 4.243 %
of XII „ in 718 „ only „ 4.131 %

The Y-amount of the left side liquid decreases:

in XI in 144 hours already with 4.663—2.651 = 2.012 %
in XII in 1679 „ only „ 4.750—3.498 = 1.252 %.

The paths VI and VII (fig. 1) have also approximately the same points 1 and 1′. In VI a membrane of collodion was used, in which a deposit of $Cu_2Fe(CN)_6$, in VII the membrane consisted of collodion only. For this reason both systems travel along different paths; besides

we see that the Y-amount in VI changes not only normally, but also anormally, while in VII it changes normally only.

It also appears from the tables that the liquids of VII change their compositions much more quickly than those of VI; we see e.g. that the X-amount of the left side liquid increases:

in VI in 1483 hours with 4.520 %
in VII in 200 „ already with 5.093 %

The paths VIII—X, of which in fig. 1 only VIII has been drawn, also have approximately the same points 1 and 1′, but differ otherwise. The membrane of VIII consisted of parchment, that of IX of collodion and that of X of collodion in which a deposit of $Cu_2Fe(CN)_6$. It appears from the tables that the liquids of IX (collodion) change their compositions quickest, those of X (collodion + deposit) are slowest. We see e.g. that the X-amount of the left side liquid increases:

in IX in 92 hours already with 2.176 %
in VIII in 194 „ only „ 1.988 %
in X in 2084 „ still only „ 1.634 %

In order to get a clearer view of the difference in rapidity with which the liquids of these and previous systems change their concentrations, we may use their Xt-, Yt- and Wt-diagrams, which the reader can draw with the aid of the tables.

TABLE VI. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.938	10.50	0					
2	43	0.383	9.970	9.743	0.268	11.1	6.8	82.1	IV	→
3	91	0.773	10.06 +	9.053	0.541	7.8	6.6	85.6	„	„
4	163	1.318	10.05	8.271	0.902	9.7	8.3	82.0	„	„
5	259	1.923	9.924	7.565	1.291	11.1	10.3	78.6	„	„
6	380	2.610	9.703	6.942	1.681	11.5	11.8	76.7	„	„
7	499	3.181	9.337	6.506	2.020	19.4	19.4	61.3	„	„
8	667	3.681	8.837	6.135	2.400	38.1	41.1	20.9	„	„
9	979	4.165	8.013	5.866	2.850	23.6	36.0	40.4	VI	←
10	1483	4.520	7.283	5.806	3.140					

SCHEME VI.

	X	Y	W
1.2	$\underleftarrow{\uparrow<\downarrow}$	* $\underrightarrow{\uparrow>\uparrow}$	$\underrightarrow{\downarrow>\uparrow}$
2.3	„	„	* $\underrightarrow{\downarrow<\uparrow}$ * *
3.7	„	$\underrightarrow{\downarrow>\uparrow}$	„
7.8	„	„	$\underrightarrow{\uparrow<\downarrow}$ *
8.9	„	„	$\underleftarrow{\uparrow<\downarrow}$
9.10	$\uparrow<\downarrow$	$\downarrow>\uparrow$	$\uparrow<\downarrow$

TABLE VII. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.721	10.59	0					
2	24	1.018	8.954	9.523	0.792	43.9	29.0	27.1	VI	←
3	49	2.390	7.877	8.121	1.832	21.0	8.7	70.3	„	„
4	74	3.123	7.283	7.496	2.404	4.7	12.7	82.5	IV	→
5	101	3.461	6.982	7.086	2.755	25.0	15.3	59.7	VI	←
6	147	4.530	5.941	5.928	3.814	16.3	6.9	76.8	„	„
7	171	4.863	5.560	5.697	4.142	2.7	8.0	89.3	I	→
8	200	5.093	5.282	5.429	4.468	3.4	7.4	89.2	„	„

SCHEME VII.

	X	Y	W
1–3	$\underleftarrow{\uparrow<\downarrow}$	$\underrightarrow{\downarrow>\uparrow}$	$\underleftarrow{\downarrow>\uparrow}$ *
3–4	„	„	* $\underrightarrow{\downarrow<\uparrow}$ * *
4–6	„	„	* $\underleftarrow{\downarrow<\uparrow}$ *
6–8	$\underrightarrow{\uparrow<\downarrow}$ *	„	$\underrightarrow{\uparrow<\downarrow}$ *

TABLE VIII. Parchment. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.933	4.484	0					
2	50	0.3744	9.397	4.132	0.4414	16.2	16.5	67.3	VI	←
3	121	0.8093	8.814	3.660	1.057	29.5	36.9	33.6	"	"
4	221	1.265	8.100	3.204	1.786	32.9	54.9	12.2	IV	→
5	360	1.678	7.247	2.739	2.641	10	12.7	77.3	VI	←
6	554	1.988	6.404	2.423	3.500	18.6	49.4	32.0	"	"

SCHEME VIII.

	X	Y	W
1—3	$\underset{\longleftarrow}{\uparrow < \downarrow}$	$\underset{\longrightarrow}{\downarrow > \uparrow}$	$\underset{\longleftarrow}{\uparrow < \downarrow}$
3—4	"	"	$\underset{\longrightarrow *}{\uparrow < \downarrow}$
4—6	"	"	$\underset{\longleftarrow}{\uparrow < \downarrow}$

TABLE IX. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.745	4.485	0					
2	24	0.779	8.395	3.653	1.370	15.5	21.3	63.2	VI	←
3	50	1.570	6.880	2.877	2.957	6.9	21.7	71.4	IV	→
4	92	2.176	5.041	2.184	4.968	2.5	3.9	93.6	I	←

SCHEME IX.

	X	Y	W
1—2	$\underset{\longleftarrow}{\uparrow < \downarrow}$	$\underset{\longrightarrow}{\downarrow > \uparrow}$	$\underset{\longleftarrow}{\uparrow < \downarrow}$
2—3	"	"	$\underset{\longrightarrow *}{\uparrow < \downarrow}$
3—4	"	$\underset{\longleftarrow *}{\downarrow > \uparrow}$	$\underset{\longleftarrow}{\uparrow < \downarrow}$

TABLE X. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.982	4.535	0					
2	96	0.5429	9.380	4.018	0.5446	26.6	25.7	47.7	VI	←
3	144	0.9557	8.816	3.552	1.095	20.9	21.8	57.3	„	„
4	264	1.196	8.482	3.329	1.411	37.4	54.6	8.0	IV	→
5	601	1.411	8.079	3.124	1.750	15.1	12.0	72.9	VI	←
6	983	1.634	7.633	2.927	2.147	27.9	57.4	14.7	III	→

SCHEME X.

	X	Y	W
1—3	$\underleftarrow{\uparrow < \downarrow}$	$\underrightarrow{\downarrow > \uparrow}$	$\underleftarrow{\uparrow < \downarrow}$
3—4	„	„	$\underrightarrow{\uparrow < \downarrow}$ *
4—6	„	„	$\underleftarrow{\uparrow < \downarrow}$

TABLE XI. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	4.663	10.57	0					
2	23	1.489	4.083	9.194	0.5199	42.7	13.2	44.1	VI	←
3	49	2.708	3.593	8.089	0.9475	32.6	9.3	58.1	„	„
4	71	3.617	3.165	7.362	1.257	10.5	0.1	89.4	„	„
5	99	4.243	2.925	6.714	1.539	1.7	5.1	93.2	IV	→
6	144	4.856	2.651	6.117	1.829	0.5	5.0	94.5	„	„

SCHEME XI.

	X	Y	W
1—4	$\underleftarrow{\uparrow < \downarrow}$	$\underrightarrow{\downarrow > \uparrow}$	$\underleftarrow{\downarrow > \uparrow}$ *
4—6	„	„	$\underrightarrow{\downarrow > \uparrow}$

TABLE XII. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	4.750	10.49	0					
2	46	0.3769	4.657	9.642	0.2642	15.7	7.1	77.2	IV	→
3	94	0.8459	4.569	8.898	0.5223	6.2	5.3	88.5	"	"
4	214	1.817	4.443	7.607	0.9353	6.0	4.9	89.1	"	"
5	358	2.795	4.365	6.745	1.209	4.7	4.5	90.8	"	"
6	526	3.652	4.263	6.205	1.384	5.6	4.8	89.5	"	"
7	718	4.131	4.250	5.992	1.445	4.2	4.1	91.7	"	"
8	1006	4.639	4.080	5.839	1.521	6.5	7.1	86.4	"	"
9	1679	5.133	3.498	5.723	1.668	6.2	15.7	78.1	"	"

SCHEME XII.

	X	Y	W
1—6	$\underleftarrow{\uparrow < \downarrow}$	$\underrightarrow{\downarrow > \uparrow}$	$\underrightarrow{\downarrow > \uparrow}$
6—8	"	"	$*\underrightarrow{\downarrow < \uparrow}{}^{*}_{*}$
8—9	"	"	$\underrightarrow{\uparrow < \downarrow}{}_{*}$

(*To be continued*).

Leiden, Lab. of Inorg. Chemistry.

Geology. — *On a new Basis of Solution of the Caldera-Problem and some associated Phenomena.* By C. G. S. SANDBERG D.Sc. (Communicated by Prof. Dr. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of November 26, 1927).

The various attempts at solving the Caldera-problem are governed, generally, by the principle of seeking an explanation:

a. of the mode of causation of a large volcanic rim enclosing, partly or entirely, a more or less flat bottom (the caldera), the diameter of which is so disproportionately large in comparison with that of the presumed magmatic conduit, that the said enclosing rim cannot reasonably be admitted to represent the primary product of eruption of the said narrow conduit; and

b. of the phenomenon of the revival of volcanic activity at or near the places of former action, after a longer or shorter period of rest.

It is the intention of the writer to indicate here only a plausible solution for the problem formulated sub *a.*, whilst accepting for the present as an empirically well established fact that mentioned sub *b.*

When considering the various studies of the caldera-problem, it appears, as far as I know, that they are governed by the following presumptions, viz.: that a volcanic cone, $b+c$, is (assumed to have been) built up round and above an eruption-centre *a.* which is, more or less arbitrarily, taken as the most likely one, and which is situated at the top of a volcanic conduit *p* (Fig. 1). Subsequently, part of the cone which would have been formed thus, i.e. the part marked *b.*, is supposed to have been destroyed, leaving

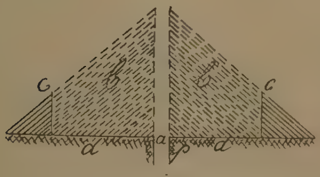

Fig. 1. Schematical section of caldera according to current conceptions. a = centre of eruption; p = Magmatic conduit or volcanic pipe; d = caldera bottom; b = vanished part of presumed original volcano; c = rest of presumed original volcano, the caldera rim.

only a (caldera) rim *c* and a *grosso modo* flat bottom *d*. In other words, the problem as it was and is posed is based on the presumption that the eruption(s) from a magmatic conduit *p* with an eruption-centre *a* caused a volcanic cone $b+c$, and its solution was and is consequently governed by the search for a plausible explanation of the disappearance of the part *b* and the causal formation of the vertical enclosing wall and the *grosso modo* flat bottom *d*, the diameter of which so abnormally exceeds that of the

presumed eruption-centre *a*. (v. HOCHSTETTER, STÜBEL, VERBEEK, DALY, DUTTON, CHAMBERLIN, WING EASTON, ESCHER a.o.).

R. A. DALY (1) expresses the current conception in the following words: "If the actually exposed "necks" of the world indicate the maximum size of central conduits, the vents beneath calderas must have cross-sections much smaller in area than the floor of the corresponding great depressions. The writer is in fact, inclined to make this the criterion for explosion craters from calderas."

The directions in which the solution of the problem have been sought may consequently be classified in the following catagories:

1. *The explosion theory.* The part *b* of the cone is assumed to have been blown away subsequent to an extraordinarily heavy explosion which is supposed to have emanated from the eruptive centre *a*, or from a point lower down the conduit *p*, part of the débris falling back into and so partly filling up the large opening thus caused, and so giving rise to a more or less flat floor.

2. *The subsidence theory.* Part *b* of the cone would have subsided along vertical or semi-vertical peripherical fault-planes, leaving only the rim *c*, subsequent to having been undermined below or above its base (somewhere near *a*) all round the conduit *p* or/and the conduits' upper extension, which we shall henceforth name the *cone-pipe*;

3. *The re-fusion-backflow theory.* Some time after the erection of the cone $b + c$, by ejectamenta from *a*, a new eruption of incandescent gas-saturated magma would have liquefied the part *b*. Subsequently to a following heavy paroxysm, part of the re-fused cone would have been blown away, the rest flowing back to deeper regions through the conduit *p* and leaving only a rim *c* round an enlarged, *grosso modo* flat "depression" *d*.

As to the *last mentioned theory*, it may suffice to refer to WING EASTON's refutation of it (2), to which may be added that e.g. the facts observed during the Vesuvius eruption (3) established that such partial re-fusion of the cone may actually occur. They were realized when the rising magma reached its highest level in the cone-pipe. Yet instead of flowing back into the conduit the incandescent magma discharged laterally, breaking through the sides of the cone. The causal formation of a, *grosso modo*, vertical inner wall, a feature which cannot be causally connected with such backflow or discharge, as WING EASTON has pointed out, did not occur.

Again, with regard to the *explosion theory* it may suffice, to avoid repetition, to refer to WING EASTON's refutation, which may be summarized in his words: "The causation of a vertical inner-wall (of a caldera) subsequent to an explosion is certainly *possible*, locally; yet it is not at all clear why this should necessarily occur, and then nearly always practically along the entire inner-circumference." (My translation from (2) p. 71.)

B. G. ESCHER's recent experiments (4), which only remotely touch the caldera-problem, have not weakened this conclusion in the least ; besides no structure comparable with that of a caldera with its characteristic features was realized in the course of these experiments.

Let us now consider the *subsidence theory*. A very sharp distinction should always be maintained between caving in, gradual crumbling or sudden collapse *(écroulement)* of the enclosing wall, the rim, with subsequent lateral *enlargement* of a pre-existent "depression" and the causation of such a large space through *a*. subsidence (falling in) or *b*. down-throw of a superstructure, presumed to have been pre-existent.

Although different in principle, these phenomena have not always been rigorously differentiated by various students of the caldera-problem. (H. RECK, R. A. DALY and others).

Now as to the conception of caldera-formation by gradual crumbling or caving in of the rim-wall, there seems to be little doubt that WING EASTON's conclusion will be generally endorsed viz. : It seems natural that even a normal crater floor (the diameter of which should be identical with that of the orifice of the conduit) may come to exceed its theoretical dimensions through downfalling, explosion or perhaps re-fusion, but that would not transform such a crater into a caldera (the diameter of the Idjen-caldera is 16 km. ; that of the Ringgit 21 km.). Besides they are characterized by very steep, often vertical inner walls (vide also v. WOLFF (5) *S*) which may attain a height of 1000 m (Rindjani) (Raoeng, Gendeng-Idjen, Tenger over 500 m). Very frequent also is the occurrence of a flat bottom sometimes called "sandsea" (Tenger, Slamet, Raoeng) [1])."

On the other hand we cannot possibly endorse WING EASTON's remark immediately following, viz. : "By these characteristics they (the calderas) are distinguished from normal craters", as will be shown below.

Since, therefore, we may discard the adequacy of the contention that caldera *formation* (not *enlargement*) may be caused by gradual crumbling or caving in of an encompassing wall, then the sole current theory which still remains to be tested is that of such causation as a product of subsidence *(effondrement)*, or of down-throw.

We have already mentioned that the caldera-structure is characterized by, among other things, a very steep, generally vertical or semi-vertical wall when not modified, secondarily, by denudation. We have also given some examples, which could be multiplied *ad libitum* (Kilauea, Barren Island Monte Somma, Knebelcaldera, Batoer, Fogo-Island etc. etc.)

This characteristic is not restricted even to terrestrial calderas, it is inherent also in lunar calderas.

The evident planetary nature of the said characteristic justifies the conclusion that it is primary and that it constitutes a genetic feature of the caldera-structure.

If this conclusion holds good it would exclude the admissibility of any

[1]) The double sound *oe* in Dutch is pronounced like the English *oo* in "poor".

of the current attempts at solving the problem, which one and all attribute the caldera-occurrence to a secondary origin.

Apart from this general conclusion, however, we shall analyse the theories sub. *a.* and *b.*, beginning with WING EASTON's.

It seems quite impossible to admit now that, before extruding vertically, the gas-channels which WING EASTON requires to produce his cells of undermining and which would have emanated, *grosso modo*, from an eruption-point *a* (see Fig. 1) would develop per se in lateral direction to such an extent as would be necessary for the production of calderas with dimensions such as are mentioned by WING EASTON himself (see ante p. 181). On the other hand it is also clear that this *impasse* in his theory cannot be overcome by transplanting the assumed point of emanation of these super-heated gas-channels to a proportionately deeper level of the magmatic conduit, as this would virtually amount to enlarging the diameter of the said conduit. In whatever way we attempt to conceive this process of honeycombing the basis of an (assumed) volcanic superstructure, this causation of cells of undermining, it must seem extremely improbable that subsequent subsidence of the (presumed) superstructure would *eo ipso* cause the remaining portion (the rim of the caldera) to be bordered by vertical walls which appear very strongly to constitute or to have constituted one continuous vertical plane. (Fogo Island St. Paul, Barren Isl., Monte Somma, etc. etc. See also p. 181. W. EASTON).

Finally it is extremely doubtful whether well established instances of such a mode of caldera-formation can be furnished.

Now, as to the assumption of caldera-formation by downthrow, it must be admitted that F. A. PERRET's studies of the Vesuvius eruptions (3) showed, among other things, that a sudden subsidence (of part of the foot of a débris-cone) may cause an opening, which might happen to be cylindrical, and to be bordered by a very steep conical wall (l.c. p. 117—118). Such a phenomenon, however, could occur only where a void was pre-existent below the subsided mass, of a size at least equal to that of the volume of the subsided mass. Now, as long as it cannot be shown that such a void must necessarily exist (or be formed in course of time) below a volcanic cone, and that the part concerned of the superstructure of the (presumed) orginal volcano must needs subside therein (sometimes perhaps by jerks *(par saccades)* and along concentric planes), so long will the calderaproblem remain unexplained on the basis of this theory, and its *quasi* solution simply means a substitution of the problem for other questionable presumptions regarding those deeper parts of our globe, of which still less is known to us with any degree of certainty.

Moreover we would emphasize that, in respect of the downthrow established by PERRET (l.c. p. 117—18), *the diameter of the incandescent magma-column then in course of ascension, in other words the diameter of the conduit, was at least equal to and most probably even larger than that of the produced conical subsidence with its semi-vertical inner wall.*

PERRET's observation cannot be invoked, therefore, in support of the subsidence theory, since the latter is based on the premise that the diameter of the magmatic conduit is *much smaller* than that of the subsided area.

On the mechanism of volcanic-cone building.

If now we may reject as untenable such conceptions of the origin of calderas as imply a secondary cause for their occurrence, then the question arises which part of a volcanic structure would be characterized, *genetically*, by those, *grosso modo*, vertical inner walls which typify among others the caldera occurrence. In order to solve this question we will examine the mode of formation of, say, a strato-volcanic-cone ; identical considerations apply to other types of volcanic cones, such as lava-domes *(Schildvulkane)*, among others. For convenience sake we shall take the form of the crater, which will generally be identical with that of the exit of the conduit at the base of the cone, to be circular, (though Askja is rectangular ; Tjiremai is oval ; the Barren Island volcano is circular ; etc.) See Fig. 2.

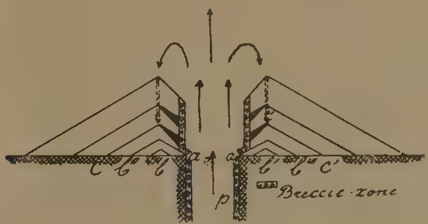

Fig. 2. Schematical vertical section of the mechanism of volcanic-cone formation.
p = Volcanic-pipe (conduit) ; aa' = eruption-centre ; bb' etc. = volcanic-cone.

We may readily conceive that a volcanic eruption emanating from a conduit p will cause the ejectamenta to accumulate round its opening aa' in the form of an encompassing cone, a rim, bb'. Under identical conditions for every part of the rim, the accumulating ejectamenta will roll or slide down under the influence of gravity and inter-friction until a state of equilibrium is reached, with its corresponding angle of inclination. Inner and outer-slope will thus be identical and in keeping with the nature of the ejected material. For strato-volcanoes this angle of slope is *grosso modo* from 30° to 40°, an inclination which is actually preserved along the outer slopes of strato-volcanoes.

In fact, supposing that the accumulation of ejectamenta and the subsequent development of the cone continues in height and corresponding base, it is clear that such may continue normally as indicated above, until its basis reaches the rim of the eruption-point a—a' (in section a—b'' and a'—b'''). Should the accumulation continue after this moment, then a new phase will have been inaugurated, as a normal readjustment of the

ejected material rolling down towards the eruption channel *a—a'* will be hampered by the pressure of the outflowing, débris-charged gas-current; whilst that rolling down along the outer slope, *tc—t'c'*, of the cone will not be subjected to such a resistance. The former will consequently be dammed up. Hence the greater thickness of the accumulated débris-strata on the cone-pipe side, as LINCK had already demonstrated experimentally. Hence, again, the lesser inclination of the cone's apex towards the cone pipe as against the outer slope.

Considering now that a volcanic cone-pipe has been built up under the influence of the factors sketched above and that the gas-current extruding from the conduit, *p*, will be vertically directed, this direction being that of least resistance, it is clear why the encompassing wall of cone-pipes will always tend genetically towards the vertical. (Chaine des Puys, Auvergne, Vesuvius, Slamat etc.). This feature is, in other words, a causal effect of the mode of formation of volcanic cone-pipes. Whereas this feature is genetically inherent in no other part of a volcanic structure but its cone-pipe; and whereas the universality of its occurrence with calderas distinctly shows that this feature is a genetic quality of these structures also, the conclusion would seem justified that the vertical encompassing wall of calderas constitutes the wall (or its remnant) of the *then* volcanic pipe. In other words, *that the caldera-capacity represents in dimension and shape, the size and form of the magmatic conduits concerned and their respective extensions, the cone-pipes, of the then volcano or volcanoes.*

Before investigating whether this preliminary conclusion finds additional support from other equally well established phenomena, we will first study more closely this vertical encompassing wall of cone-pipes.

ESCHER's remarkable experiments (4) (Pl. 7) demonstrated the mode of development of such a channel in a homogeneous cover when pierced by a vertically directed gas-current. It is true that a volcanic cone does not represent the body of a pre-existent mass which covered the eruption-point and that strictly speaking, the cone-pipe is not as a rule pierced through a pre-existent cone; on the contrary the latter is built up all round the former. ESCHER's basis would consequently seem to be false. Yet in considering the mode of formation of volcanic cones as detailed above, it is clear that in its results there is no difference between the piercing of a pre-existent complex of strata by a gas-current and the keeping open of a channel by a débris-laden gas-current which would be covered by the ejected material but for the clearing action of the current. (The cone-pipe walls of lava-volcanoes *(Schildvulkane)* encompassed by more homogeneous material, yet formed in an identical way, are also vertical (5) p. 454 ff.).

If however the complex of strata through which such a gas-current is extruding should be heterogeneous, i.e. constituted of irregularly alternating more and less resistant strata (Kimberley pipes), then cavities may actually be hollowed out in the *grosso modo* vertical wall of an eruption channel (conduit or/and cone-pipe) by the greater eroding effect of the action of

the current on strata of less resistance; which is clearly demonstrated in ESCHER's experiments (4) p. 70—71. Yet it should be noted that such deflections from the normal will only occur, theoretically, subsequent to much heavier explosions (gas-extrusions) than those which caused the erection of the cone-pipe. In every other case, in fact, the erosive effect of the ejectamenta-laden gas current, if any, may be neglected, as the material constituting the con-pipe arrived *in situ* at its state of equilibrium under the influence of gravity *and* that of the pressure of the gas-current. Now, as the former remains unchanged, it is clear that the state of equilibrium will remain unaffected also, unless the latter (gas-pressure) is considerably augmented (a considerable diminution of the gas-pressure may cause a local down-fall of material, as we shall see). Finally we would draw particular attention to a very remarkable observation made by PERRET (3) (p. 113) to the effect that a magma rising in a cone-pipe will tend to fill up any such deviations from the vertical in the wall of the cone-pipe by a process which PERRET describes as "plastic lining". Schematically we could represent the vertical section of a cone-pipe with its orifice, the crater, *during* a constant ejectamenta-charged eruption as given in Fig. 2. At the close of such an eruption the ejectamenta accumulated all round the crater will be in a state of labile equilibrium which may be disturbed by the slightest shock, an air-vibration even, as PERRET was able to establish. Thus the enormous quantities of latent energy accumulated in the masses surrounding the orifice, the crater, when rendered kinetic, will tend to establish a state of equilibrium, thereby sometimes causing huge avalanches to crash down to the bottom of the cone-pipe, and thus enlarging the *crater* and steepening its inwardly directed slope, to an angle greater than that corresponding with the angle of the slope of a normal débris-cone of the material converned, an angle which is preserved in that of the outer slope of a volcanic cone (3) (p. 99 ff and Fig. 63, p. 103).

Before closing our study of the eruption-channel, i.e. that of the magmatic conduit and its extension, the cone-pipe, we will point out that its mode of formation will tend to produce at least three principal zones of less resistance, to wit: one situated more or less centrally at or near the vertical axis of the eruption-channel; a second along the border of the said channel, which we will call the inner-peripherical zone; and a third more excentric still, which we will call the outer-peripherical zone. The existence of these zones is strikingly manifested in nature by the common occurrence of more recent points of eruption situated peripherically as well as centrally with respect to older channels (3) (p. 19, Fig. 3; (7) Kawah Ratoe of Tang-kubang Prahoe; G. Tjiremai, a.o.). The phenomenon is also manifested by a marked tendency of primary fumaroles to occur centrally or else inner- or outer-peripherically and which is beautifully illustrated in the structure of the G. Pajang (Batoer-Complex), the remains of which disclose fine sections, vertical and horizontal. KEMMERLING established that within the, *grosso modo*, vertical cone-pipe, the solid central core is separated from its

clastic mantle, the cone, by a zone of breccias enclosing the said solid core.

This remarkable phenomenon of the re-occurrence of younger eruption-points, centrally or and peripherically situated in relation to an older one, which is as general an occurrence with volcanic cone-pipes as with calderas, strikingly discloses yet another close similarity between these two phenomena. A few examples, which may be multiplied ad libitum, of peripherically situated younger eruption-points of calderas are the Fogo Island, G. Batoer (with its G. Abang), the Knebel-caldera (Rudolf-crater and the S. E.-craters), the Piton de la Fournaise (Reunion) (9) (p. 263) with its peripherically arranged fumaroles; whilst more centrally situated younger eruption-points are exemplified in Fogo-Island (10) (p. 29, Fig. 32), Barren Island (11), G. Batoer (the active cone), Vesuvius, etc. etc.

Moreover we find both types abundantly represented among the lunar calderas.

On caldera-capacities considered as the original openings of older volcanic cone-pipes.

Having indicated the similarities between calderas and volcanic cone-pipes and having shown that at least one of the qualities they possess in common, that of the vertical inner wall, is a genetic feature of the latter and most probably also of the former, we will now examine:

1. How, if at all the genesis of calderas could be explained rationally on the basis of the assumption that the phenomenon differs from that of a volcanic cone-pipe in relative size alone;
2. Whether examples of caldera-formation in the manner assumed by us are known; and
3. Whether and how the actual remains of caldera-structures furnish a rational basis for reconstructing their history and mode of development.

Sub. 1. *Simple Calderas.* If the conception of the word *caldera* includes all those "depressions" of volcanic origin which are enclosed in toto or partly by a cone-shaped débris-mantle characterized by a steep, semi-vertical or vertical inner wall and a normal inclination of its outer slope, then the word would also include those volcanic "depressions" within which no younger eruption-point could be established. We shall call this kind of calderas *simple calderas*, the type of which is represented by, for instance, the Ngorongoro-caldera in East Africa, with a diameter of 17 to 22 km (personal communication of H. RECK), the Askja, of some 9 by 9.5 km [1]) and the Knebel-caldera (Island) of some 4.5 by 2.5 km in diameter.

Now, whereas the diameters of eruption-channels and their corresponding cone-pipes may vary from a few meters and even less (adventive craters, hornitos etc.) to 200 m (Vesuvius before 1906) (3), 400 m and 500 m

[1]) These dimensions, derived from his topographical map, do not seem to agree with H. RECK's estimates of its size, which he takes to be about 55 km² (p. 45).

(Vesuvius after 1906), 5.6 km (Kilauea) (5) (M. Loa, etc.), it seems extremely difficult to understand why the conception should be inadmissible. that the diameters of cone-pipes have been larger in the past than the average of those we know now. and, consequently, why the calderas above mentioned could not be the remnants of the cone-pipes of volcanoes. On the contrary, the very absence of smaller eruption-points within such calderas, although it could never be invoked as a direct proof, may plausibly be explained by such a conception, which would readily solve a haunting enigma.

In the case of the Askja-caldera, moreover, the above conclusion is strengthened by the structure and nature of the encompassing wall, the Dyngjufjöll, more especially by the typical peripherical arrangement of the smaller calderas, the so called "Lava Plateau", in the north east, and the Knebel-caldera in the south east of Askja, and again by a repetition of this mode of occurrence of younger eruptive channels round the periphery of the Knebel-caldera. In fact here we find the Rudolf-crater, the south eastern craters and southern fumaroles grouped in a way precisely similar again to that of secondary cones round older volcanic vents (see below) (6) (12).

THORODDSEN (12) (p. 198) considers rightly, in our opinion, the Dyngjufjöll as a remnant of an ancient strato-volcano. This view of the nature of the Dyngjufjöll finds support in its structure of alternating strata of lavas and clastic material and in their directions of slope ; yet, that the enclosed Askja-caldera would have been produced subsequently, and by down-throws of part of the upper structure along vertical fault-planes, we cannot admit.

Sub. 2. *Composite calderas.* Under this head we comprise such calderas as have or have had one or more indubitable eruption-points, situated more or less centrally or peripherically to the encompassing wall (the inner wall). (Barren Isl., Tenger, Vesuvius, etc.). This definition would virtually comprise also forms of calderas with eruption-points so excentrally situated that they actually occupy an outer-peripherical position. Such is the case e.g., as we previously pointed out, in the Askja- and Knebel-calderas which we discussed sub. 1 as *simple calderas*.

Now, if we admit that the caldera-space constitutes the space within the remnants of the cone-pipe of an older volcano, it implies that the composite-calderas would have been formed by a *process of filling up*, to a more or less degree, of such older cone-pipes by the products ejected by one or more succeeding younger eruptive cones.

The exact conception of the mechanism of such a process may best be realized by a description of such a caldera-formation which was actually witnessed and minutely registered in all its phases of development. It shall at the same time be an answer to the question posed sub. 2.

The example we shall choose is that of Vesuvius, which, with its Somma encompassing a still active eruption-point, represents the classical example

of a composite-caldera. We choose this particular example because we know of no other strato-volcano, whose history has been so accurately registered during such a long period as has that of Vesuvius. Among the records we would specially mention those of the observations made by PERRET (3) of the eruptions of 1906 and 1913—'20. Their accuracy, minuteness and instructiveness and the vividness of their description permit one to follow the development of the phenomenon step by step and support, while severely testing, our contentions.

First of all we wish to place on record that both MERCALLI and PERRET (l.c. p. 14) seem convinced that the encompassing Somma-wall is nothing but the inner wall of the older Somma-volcano. DANA arrived at a similar conclusion with regard to the inner wall of the Kilauea-caldera [1]). Yet, so far as I know, neither of them nor any one since seems to have realized that these conclusions with respect to the caldera-walls of Kilauea and Vesuvius might contain the solution of the caldera-problem in general. Let us now study the history of the present eruptive channel of Vesuvius from the time immediately preceding its eruption of 1906.

The crater, which is the *orifice* of the cone-pipe, situated a little excentrally within the encircling Somma-wall, was then some 180 m in diameter, while that of the Somma-caldera measured some 3.5 km.

When we consider the observed phenomena bearing more specially on our subject, we find that the diameters of the crater and of the cone-pipe were enlarged respectively to 1000 m and 400 m by the 1906 eruption.

The passage from the lower-rim of the *crater* to the top of the cone-pipe-wall was sharp, not gradual (l.c. p. 98) ; the inclination of the crater was 45° and that of the cone-pipe consequently steeper ; the depth, from the top of the cone to the floor or the crater, measured 400 m so that the height of the steeper cone-pipe-wall above the crater-floor must have been some 100 m. These data characterize conditions actually existing in 1909, i.e. after three years of denudating activity. That the cone-pipe wall was perpendicular or nearly so, during and even years after the 1906 eruption, is testified by the various photos (3), and still more conclusively by the shape of the gas-column, 13 km in height, which was ejected through the cone-pipe during the eruption. In fact the apex of this inverted cone barely measured 20° so that its conduit must have had an inclination of at

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[1]) In view of the greatness of the discharge in 1823, — so undermining, owing to its extent, as to drop abruptly to a depth of some hundreds of feet the floor of the crater leaving only a narrow shelf along the sides, — we reasonably conclude that at that time the lava-column beneath the floor was of as large area as the Kilauea pit itself, — or nearly seven and a half miles in circuit. We may also infer that, immediately before the discharge, wherever there was a lava-lake, the liquid top of the column was up to the floor of the crater, and elsewhere not far below it . . . . When the floor of the pit fell at the discharge in 1840 it was not thrown into hills and ridges, as it might have been had it dropped down its four hundred feet to solid rock in consequence of a lateral discharge of the lava beneath; on the contrary it kept its flat surface, thus showing that it probably followed down a liquid mass, that of the subsiding column of lava. (13) pp. 151—152.

least 80°. This phenomenon therefore furnishes a striking corroboration of our theoretical deductions from a study of the mechanism of cone-pipe-building which led us to the conclusion that the walls of volcanic cone-pipes will causally tend towards the vertical. In fact these extremely pointed inverted gas-cones, far from being an accidental mode of occurrence of this or previous Vesuvius-eruptions, are characteristic, as is well known, of certain types of gas-eruptions known as volcanian eruptions. This kind of eruptive manifestation, moreover, does not pertain to any specific type of volcano but may and often will occur in the course of any period of activity of any volcano.

PERRET does not describe the condition of the crater-floor immediately after the eruption of 1906 ; three years later, however, it appears that on it several débris-cones, derived from avalanches of crater- and cone-pipe-material, had accumulated against the steep wall of the cone-pipe. This condition remained practically unchanged until 1913. A new phase then announced its approach by an intensified activity of the magma.

Subsidences and magmatic absorption of parts of the crater floor and in particular of the foot of a débris-cone, and alternating formation and subsidence of eruptive conelets were the external signs of an incandescent magma rising in the conduit, the typical glare of its liquid surface manifesting itself on July 8 [1]).

It should be remembered that a conical depression (100 m in diameter and 20 m deep) with a semi-vertical wall was consequently formed in the foot of a (the south-western) débris-cone by subsidence. From its centre a volcanic conelet built itself up, its base closing over the lower part of the said depression. At the end of October lava began to flow out from the top of the conelet, gradually filling up first the enclosing depression and then the entire cone-pipe up to its junction with the lower rim of the crater.

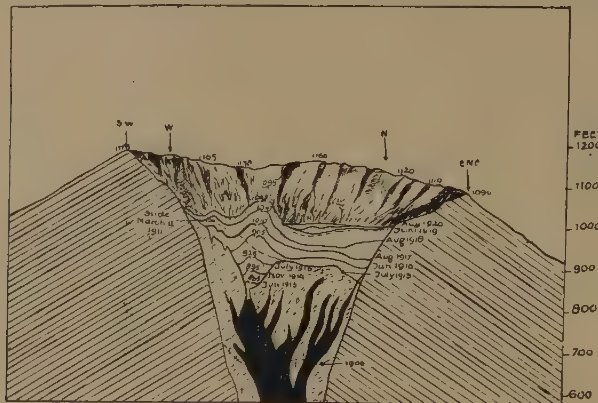
"The rising lava soon formed an eruptive conelet, and from this time "onward, up to the day of writing (1921), the entire course of events in "the external activity of this volcano has been characterized by an almost "continuous process of crater-filling activity (Fig. 61, our Fig. 3), through "superposition of material erupted explosively and effusively from the "eruptive conelet and adventitious vents."

Thus PERRET *(l.c. p. 119 ff.) summarizes his exact observations of a most remarkable phenomenon by the development of which the cone-pipe of Vesuvius of 1906 was converted into a miniature caldera, that of 1921, by a filling-up process identical, no doubt, with that which at one time (An. 79 b. C.?) converted the older Somma-pipe into the caldera-wall of the younger Vesuvius of that time.*

All the typical features characterizing terrestrial calderas equally

[1]) The records of the very accurate observations during the entire course of development of this phenomenon place it beyond any doubt that the ascent in the cone-pipe of the incandescent lava (magma) with its highly corrosive vapours and gasses occurred excentrally (semi-peripherically).

characterize the actual minute younger Vesuvius caldera ; the abnormal proportions between the diameters of the younger eruption-channel and of



DRAWING BY MALLADRA.

Fig. 3. Schematical representation of the growth of the eruptive conelet and the resulting process of filling up the crater of Vesuvius. Composite caldera-formation. Drawing by Malladra; copied from (3) Fig. 61; section of Fig. 4.

FROM MALLADRA. MAP OF CRATER FLOOR ON DECEMBER 31, 1920.

Fig. 4. From Malladra. Map of crater floor of Vesuvius on December 31, 1920, showing a perfect composite caldera of fair size. Copy (3) Fig. 62.

its caldera are of an order identical with other, larger terrestrial composite calderas ; the *grosso modo* vertical encompassing wall is present also, even

though it is now almost entirely hidden from view by the filling; the flat bottom, generally speaking, was caused and maintained by the extrusion of highly liquid lava; finally, if the lava extrusions had been succeeded here by ejection of loose material (ashes, lapilli, etc.), the characteristic "sandsea" of Dutch East Indian and other volcanoes would certainly have been present in this case also. Thus this "sandsea"-phenomenon would likewise have found a ready explanation.

It is now necessary to test our contention, that the caldera space is nothing but a remnant of partly filled up older cone-pipes, on some other well known calderas.

On the probable mode of formation of certain calderas.

If the younger eruption-channel of Vesuvius had extruded centrally in the resent Vesuvius-caldera above described, just as it occurred in the G. Raoeng (7) (p. 58, Phot. 18), then we should have had a form of caldera comparable in every respect with that of Barren-Island and similar calderas.

The crater of the Slamat (7) (p. 35 ff and Phot. 8) presents a more composite caldera. "The crater proper (diameter over 400 m depth 228 m) is situated in the south western part of the cone and is enclosed on its north eastern side by some three crater-rims. Between the most northern and the central one a great plane or sandsea spreads out, strewn over with numerous bombs" (l.c. p. 38). Evidently we have here three inter-telescoping cone-pipes mutually tangent in the south west, the older ones of which are enclosing their successors (l.c. p. 39) in consequence of a successive displacement of eruption-channels in a south western direction. The result of such an occurrence was that the cones of the younger eruption-points could only develop individually in a north eastern direction and hence it is only in that direction that we find their remnants, namely the individualized encompassing walls and the sandsea (Fig. 5).

To nobody, we are sure, would the idea occur that the walls of this caldera, though a miniature of its kind, might be the product of down-throw, subsidence or magmatic fusion. Nobody could doubt that these walls are the remnants of older cone-pipes of which only the oldest was levelled a little or hollowed out, probably by the erosive and denudating action of (a) gas current(s). Yet, apart from its total size, this kind of caldera differs from the occurrences generally designated by that name, only in so far as the diameters of the older and younger cone-pipes differ but little here, which implies that the free development of the more recent was hampered. Still in essentials there is no difference whatever between this and other calderas.

The crater of the Sendoro (7) (p. 44 and Photos 10 and 11) presents a similar caldera within which "the remains of a younger cone encompassed

by an older wall" is visible so that this crater again reproduces the live type of caldera with all the characteristic features.

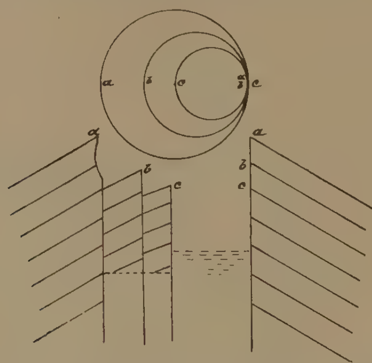


Fig. 5. Schematical section of the Slamat cone-pipe and crater shewing its caldera structure. Compiled by me after the description and aeroplane photos from (7)*pp* 35*ff* and Photos 8—9). *aa* = section of oldest cone-pipe; *bb* and *cc* of younger pipes with sandsea between *a* and *c* on the left.

It would not be difficult to give any number of other examples of typical caldera-craters (or crater-calderas) and it was in fact the comparative study of crater-types, especially the East Indian, and calderas, which led us to the conviction that these phenomena are essentially alike and differ only in the relative size of the eruptive channels.

Let us, finally, study the mighty caldera of the Batoer-complex, the Molengraaff-caldera on Bali, with its axes measuring 13.5 km in a N.W.—S.E. and 10 km in a S.W.—N.E. direction (8). Its floor lies at an altitude of 1000 m the highest points of its rim at 1745 and 2152 m and the lowest at 1267 and 1336 m respectively. Fig. 6, compiled after a topographical sketch by KEMMERLING, supplemented by my own observations, gives the main features of the caldera, a section of which is also given by KEMMERLING (8) [1]).

The vertical inner wall of this, the Molengraaff-caldera is very remarkable indeed. It closely follows the rim of the caldera between the points 1745 (G. Penoelisan), 1371 (W. side), 2152 (G. Abang), and 1270 except where it is interrupted by downfalls or hidden from view by débris-material, as is specially the case, e.g. E.-S.E. and N.-N.W. from the point 1371. Within this huge enclosure, which we will henceforth call the

[1]) This section is for its north western part incorrect, in so far as it does not show the vertical wall of the Molengraaff-caldera there, although parts of it may still be recognized here and there between the points 1745 and 1371, however much destroyed it be by avalanches. The vertical inner border of the lower plateau, at the south east side of the section, is most probably a remnant of the wall of the Molengraaff-caldera (the "outer-wall"), and KEMMERLING seems to concur with this interpretation of the nature of this feature (l.c. p. 61).

outer wall, in contradistinction to another, powerful yet smaller caldera-wall enclosed by it, we find the remnants of the latter, as another vertical,

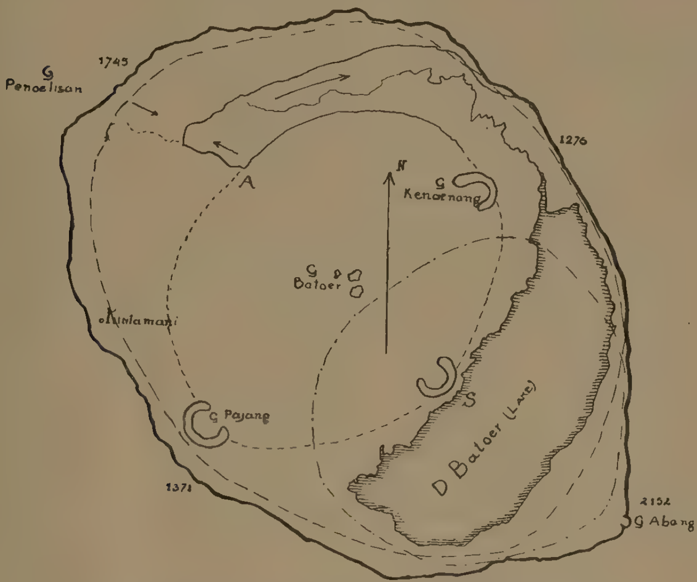


Fig. 6. Main lines of the Batoer-complex with its Molengraaff-caldera; after a topographical sketch (8) and personal observations.

roundish caldera-wall, the diameter of which must have measured some 7.5 km From a point A. marked on the sketch, eastward to a point close to G. Kenoenang, this steep *inner wall,* towering up vertically to a height varying between 200 and 300 m above the floor of the caldera, is scarcely modified. Its extension in the opposite direction, towards G. Pajang, is very mutilated and in parts almost destroyed, whilst that part which probably extended between G. Pajang, the eruption point marked S. and G. Kenoenang, has completely disappeared.

This steepness, *casu quo* verticality, of the inner-walls of the Molengraaff-caldera and its inner caldera, suggest that they probably constitute the remnants of the respective cone-pipes and their conduits.

Moreover, the said plateau, a sand-sea *optima forma,* points in the same direction, as it evidently constitutes a remnant of the cone of the inner caldera.

In fact it slopes outward and the general direction of its incline is towards the E. (vide its drainage) i.e. in exactly the same way as the rim of the Molengraaff-caldera between G. Penoelisan (1745) and the point 1276. Both the direction of these inclines and their parallelism may safely be taken as a result of the influence of current winds on the ejected

material during the building processes of the respective cones. On the other hand, it would be inadmissible to interpret this concurrence of inclinations and the outward direction of the slope of the plateau, as the necessary results, i.e. causal effects of subsidence, downthrow, explosion or fusion.

Finally, we find peripherically arranged along the border of the ,,inner-caldera'' first G. Pajang, than probably the eruption-point S. and finally G. Kenoenang, on the very zone which would have marked the zone of least resistance of the "inner volcano", namely on the zone situated between the solid core and its encompassing cone.

Since the erection of the inner-volcano above mentioned, it would seem that another eruptive channel forced its way, destroying or lowering not only the south eastern part of the *inner wall* but blowing away also part of the south eastern wall of the Molengraaff-caldera. Proof of this we find in the remaining half of G. Abang which is outer-peripherically situated in the *outer wall*, and which volcano was most probably halved in precisely the same way as was the *Piek of Rakata*, (on the outer periphery of Krakatau) during the well known Krakatau eruption of 1883.

This complex of Batoer-eruption-channels, localised within the encompassing *outer wall*, would thus have formed a twin-volcano similar to Tangkoeban Prahoe, which latter also must have been enclosed within an encompassing wall (14) (p. 732) the remnants of which may still be traced over a distance of some 15 km from G. Nanggarak (S. of Tjisaroea) over G. Lembang and Pr. Malang as far as Pr. Pangoekoesan (7) (p. 73). The present stage of activity in the history of the Molengraaff-caldera would have been inaugurated, then, by the eruptions of the peripherical eruption-points, G. Pajang, *S*, and G. Kenoenang, and finally by the eruptions of the centrally situated, still active G. Batoer, which is continuing to fill up the Molengraaff-caldera to the present day.

In the foregoing we have demonstrated by a few examples, which could be multiplied *ad libitum*, that the caldera phenomenon in its various aspects may be plausibly explained on the assumption that the vertical wall encompassing the caldera occurrence is nothing but the wall of an older eruption-channel, its cone-pipe or crater, or their remnants. Thus a composite caldera (see above) would be the remnant of an older eruptive channel filled up in part or *in toto* by younger eruption products, emanating from an inner, more or less considerably reduced, younger channel during a period of locally renewed magmatic activity.

The mechanism of such a filling-up process, resulting in the production of a miniature composite caldera, could be followed step by step during the Vesuvius-eruption of 1913—1922 (vide Figs. 3 and 4).

Should this conception of the nature of the caldera-phenomenon be correct, then we would have to conclude e.g. that the intensity of terrestrial volcanism has been dimishing over a large part of our globe at least since Tertiary times and perhaps since an earlier period. Whether such diminution

dates from still earlier geological times, whether it comprises terrestrial volcanism in general, whether it is only of a local, relative or/and intermittant character, are questions the answer to which would require more extensive and detailed studies all over our globe.

It is my personal conviction that aeroplane-photography, if appropriately and systematically conducted, may render considerable service towards the solution of these questions and to that of magmatic activity in general. We do not consider it unlikely that in this manner the presence of terrestrial calderas with dimensions equal to or even surpassing those of the moon may be located.

On zones of secondary eruptions and on the causes of displacements (migration) of eruption-channels.

In the foregoing we have already touched upon (p. 185) another phenomenon pertaining to cone-pipes and calderas alike, which, consequently again points to an identity in substance of these volcanic phenomena. Considering moreover the universality of its character and the identical way of its mode of occurrence in the case of both cone-pipes and calderas, it constitutes a strong indication that the phenomenon may be a genetical feature, inherent in the formation of these volcanic structures.

We are alluding to the marked tendency of secondary eruption-channels to occur e.g. on the periphery of older eruption-channels. This phenomenon occurring universally, as is well known, in all kind of volcanoes (Hawaian volcanoes, Vesuvius, Bromo-Segoro-complex etc. etc.), pertains equally to calderas (Molengraaff-caldera, Fogo Island, Vesuvius-Somma-complex Askja-Knebel-group, etc. etc.).

Although secondary (i.e. younger) eruption-points may occur also in other places and from other causes, we will restrict ourselves here to the study of this particular mode of occurrence.

The occurrence of secondary eruption-points has hitherto been currently, explained, as a causal expression of the influence of fissures or faults radially or periclinally directed, whilst, inversely, the presence and *quasi* mode of occurrence of these secondary eruption-points is often advanced as the only vindication of the assumed existence of such fissures and faults (14) (7) (3) (5, p. 415, etc.). We do not wish to deny that eruptive occurrences may have been provoked by the presence of fissures and faults ; yet the idea, that the latter constitute the main (only ?) and primary cause of the former, we cannot admit unreservedly.

In fact, when considering the mode of arrangement of these secondary eruption-points with respect to the crater or caldera concerned, it soon becomes obvious that we may distinguish three main groups, i.e. : 1. That in which they are more or less centrally situated ; 2. that in which they are arranged along the inner side of the craters or calderas (inner-peripherical); 3. that in which they are arranged on or outside the cone-rim of craters or calderas (outer-peripherical).

The natural section of G. Pajang shows that the magmatic core, the filling of the cone-pipe, is separated from its clastic cone by a zone of breccias. PERRET (3) observed a similar occurrence at Vesuvius, a form of which he designated as the "plastic lining" of the cone-pipe wall. Which ever form the phenomenon may effect it may reasonably be expected that the cooling effect of the wall on a lava-column rising in its conduit (or cone-pipe) will tend to cause a zone of discontinuity between such column and its encompassing wall and that such tendency will be accentuated by the shrinking of the lava subsequent to its consolidation. Its formation is consequently genetically inherent in the mode of formation of a volcanic cone. (N. J. M. Taverne arrived at a similar conclusion in 1925).

The problem of the occurrence — so frequent — of secondary eruption-points and fumaroles arranged inner-peripherically, would thus find a ready solution in the presence of this zone. Moreover it would explain the peculiar and marked tendency of more recent eruptive and fumarolic action to (inner-) peripherical migrations, a phenomenon which is so common in all volcanic massives (see Fig. 7).

How are we to explain, however, a similar tendency of magmatic activity

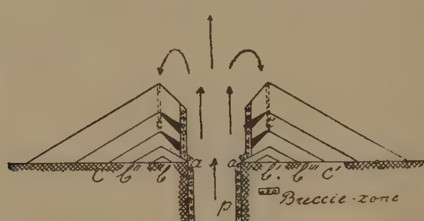


Fig. 7. Schematical vertical section of a volcanic cone shewing that the tendency of younger eruption-points (fumaroles etc.) to peripherical migration is a genetic quality, causally inherent in the mode of formation and subsequent structures of volcanic cones.

to that super-excentrical displacement of its successive sites which we have qualified as outer-peripherical ?

When studying the structure of a volcanic cone more closely, we find that the zone of less resistance above described is bordered along its outer periphery by the clastic cone which, from this zone outwards, *is built up in anticlinal fashion.*

It is clear that the highly tensioned magmatic gasses and vapours, highly corrosive at that, will tend to discharge along this zone of less resistance and thus to penetrate into these clastic anticlinals, the roots of which are open along the said zone. Arrived at the tops t, t', etc. of these anticlinals (which are again arranged, *grosso modo,* along a vertical concentric plane going through the top-rim) these vapours will meet with an extra-resistance when arriving up against the downward flank of the anticlinal. They will consequently collect at these several tops and their tension and

corrosive capacity will tend to make them drill their way vertically through the covering masses, causing their discharge at last along and beyond the top-rim of the volcanic cone, i.e. *outer-peripherically*.

It is evident, therefore, that the outer-peripherical arrangement of secondary eruption-points and fumaroles, which pertains inherently to calderas and smaller volcanic cone-structures alike may be readily explained from the mode of formation of a volcanic cone surrounding its cone-pipe.

A striking confirmation of our contention as to the identity of cone-pipes and encompassing caldera-walls is again furnished by the Batoer-complex. A close study of the section of this caldera (8) discloses in its S.E. corner a remnant of a vertical inner wall, which is the inner border of a remarkable plateau situated at the foot of the bisected cone of G. Abang, and which in its turn constitutes part of the *outer wall*, i.e. of the Molengraaff caldera-wall.

In respect to this main-wall of the great Batoer-caldera, the bisected eruption channel of G. Abang [1]) is situated outer-peripherically whilst the plateau extends between a remnant of the original encompassing wall of the said Molengraaff-caldera and the bisected eruption channel. Hence we find that the slopes of the said plateau, which evidently represents a remnant of the old Batoer-cone, are directed inwards *and* outwards in anticlinal fashion.

It would seem that we have now sufficiently explained and founded our contention about the real nature of the caldera phenomenon, to which subject we intend to return shortly in a second communication.

Finally we have to offer our sincere thanks to both Dr. P. TESCH, director of the Geological Survey of the Netherlands at Haarlem, and to Mrs. E. A. VAN OOSTERZEE—BEELAERTS VAN BLOKLAND at the Hague for their kind assistance in the execution of the drawings which accompany the text.

The Hague, September 1927.

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[1]) It would seem that section and description do not correspond completely. The former is in contradiction also with my personel observations. Apart from my rectification in the foregoing regarding the interpretation of the north western and western wall of the Molengraaff-caldera, it should be noted that the wall below the top of G. Abang is very nearly vertical. This fact seems to find confirmation in KEMMERLING's own descriptions appearing on pp. 51 and 61 of (8).

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Geology. — *On a new Basis of Solution of the Caldera-Problem and some associated Phenomena.* By C. G. S. SANDBERG D.Sc. (Second communication.) (Communicated by Prof. Dr. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of February 25, 1928).

In our previous communication (17) our consideration of the caldera-problem was based, more especially, on the mode of occurrence of the phenomena in strato-volcanoes, and on that of the migration of eruption points. Accordingly we tested our conclusions on well established facts in the history of Vesuvius, that volcano being the most carefully observed and described specimen of its type.

We now shall use the results of the studies of the classical type of lava-volcanoes *(Schildvulkane)*, the Hawaiïan and specially Kilauea, as testing material for our contention regarding the real nature of the caldera phenomenon, this lava-volcano, like Vesuvius, having been very closely observed during several decades.

Halemaumau, its still active eruptive centre, is situated in the south-western part of the great Kilauea-caldera, the vertical wall of which encompasses an area of some 5 by 3 km in diameter.

Inner-peripherically situated remnants of former eruption-channels of similar order may be located, S. of Volcano-House, at the N. E. corner of the caldera and in its south eastern part, the Sulphur Bank. Also, in 1888 (see WILKES and BRIGHAM's map [13] (Pl. IX)) two smaller specimen were situated on the edge of the then "black ledge" and two others occurred a little nearer to the north western part of the caldera-wall (Fig. 1 ; i.k. and l.m. respectively). These four eruption-points and the

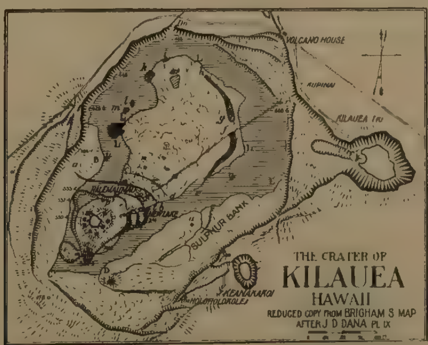

Fig. 1.

lava-dome, to the S. of the last two mentioned, trend in a direction parallel to the north western part of the Kilauea-wall. Such concentric arrangement of eruption-points is repeated in the "black ledge" round Halemaumau (p.q.r.s.), where they occur *outer*-peripherically in relation to the present eruptive channel of Halemaumau and at the same time *inner*-peripherically in relation to a former, wider eruptive channel, the remnants of which surround the present one and the intervening black ledge.

Outer-peripherically arranged round the Kilauea caldera-wall we find Kilauea-Iki, a "depression" to the W. of it and Keanakakoi, and this phenomenon is repeated round Halemaumau by the mode of arrangement of the so-called "New Lake", a "depression" to the N. W. of it, and a similar one on the extreme S. W. of the old cone-pipe-wall of Halemaumau above referred to.

As further examples of main orifices surrounded by peripherically arranged secondary eruption-points we may mention Mauna Loa, Kea· (19), the volcanoes of the Samoa Islands etc. etc. (20).

DANA's contention that the vertical wall encompassing the Kilauea caldera is nothing but a remnant of the cone-pipe-wall of the then Kilauea volcano is now accepted by JAGGAR, PENCK, ARN. HEIM, PERRET, among others, in so far that they agree that "Halemaumau is a volcano in a volcano", as ARN. HEIM expresses it (21). Curiously enough, however, the latter still qualifies this Kilauea-wall as that of an "Abbruchskrater", a "cratère d'effondrement", the down-throw of which would have resulted from undermining through fusion of its support (l.c. text to Pl. I). This contention seems in flagrant contradiction with established facts of Kilauean history and appears to be based, among other things, on an erroneous way of extrapolating observed phenomena of secondary moment.

In order, to make the gist of my argument clearer, I reproduce DANA's combined section of Kilauea, showing the successive changes in the form of its crater (cone-pipe) during the period 1823—1886 [13] (p. 127) (compare also his maps, sketches and descriptions). This section clearly shows e.g. how, in the course of time, the crater of Kilauea narrowed down to that of Halemaumau.

A study of DANA's section and descriptions clearly shows that ever since

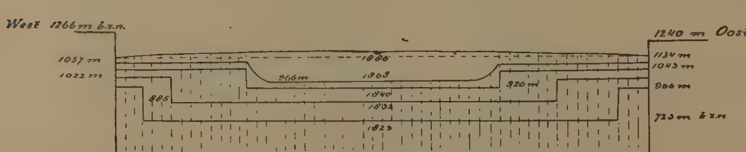

Fig. 2. Schematical combined section of the Kilauea-Caldera showing its mode of development subsequent to its principal eruptions during the period 1823—1886. After J. D. DANA. The broken vertical lines, added to the original by me, represent the directions of the dividing planes between successive accretions. Those between successive depositions, by overflows, have not been indicated to avoid crowding.

the 1823 eruption (and probably even before that of 1790) the absolute height above sea-level of the top of the caldera-wall of Kilauea, as well as its shape remained unchanged; i.e. during that period there occurred no elevation, subsidence (downthrow), or caving in of the caldera wall of sufficient importance to support the contention that the occurrence of this wall is of secondary origin, as is implied by current theories. In 1823 a narrow terrace extended (300 m below the western top-rim of the wall) all along the inner side of the caldera-wall of Kilauea. It was the black ledge of that time, the vertical inner wall of which enclosed the then crater-floor, the bottom of the "lower pit".

Now, how was this, and how were the younger terraces successively formed?

By down-throws or subsidences? By outblasts? or in some other way?

It is clear that a positive answer to these questions, if it could be given, might be decisive as to the origin of calderas.

We cannot affirm, on the evidence of direct observation at least, that the terrace of 1823 was actually formed in some other way, however much we may be justified in doing so on the basis of anology.

In fact, if this and subsequent terraces were formed by downthrow, collapse or out-blast, then any such process of presumed caldera-formation would necessarily have caused:

1. an outward displacement of the caldera-wall or (and) of that of the enclosed cone-pipe-wall; i.e. a permanent *enlargement* of either or both;

2. part of the former mass surrounding the eruption-channel, i.e. part of the (presumed) original cone and (or) its floor, to occupy a lower level after the supposed event than it did before (compare fig. 1 (17)).

Now, when considering how the Kilauea-caldera and its enclosed eruption-channel, the lower pit, fared after 1823, we find that no outward displacement of the Kilauea-wall worth mentioning has taken place, nor, we repeat, any material change of its altitude above sea-level, in spite of various eruptions since 1823. It is therefore most probable that the then terrace below the wall-rim was no more a product of subsidence, downthrow or outblast than are those of later formation. In fact, not only did no enlargements of the caldera space or of the magmatic conduit occur, subsequent to the eruptions between 1823 and 1886, *but the contrary actually happened, in so far that the capacity of the latter, that of the "lower pit", narrowed during each eruption until it was finally restricted to the dimensions of Halemaumau, the present eruptive channel.*

Correspondingly the surface of the black ledge, enclosing the channel, broadened and also assumed a **higher** *level than that of its predecessor, because of the deposition of fresh ejectamenta on its former surface.*

Thus, the surface of the caldera-floor acquired its maximum height and development subsequent to the eruption of 1886. It then extended as a slightly convex plane, over the entire space enclosed by the encompassing

wall of Kilauea, at an altitude which was only 132 m below that of its western rim, a formation exactly like those of Askja, which is enclosed by Dyngjufjöll, and of Ngorongoro (East Africa). Consequently, if volcanic activity at Kilauea had ended then or had remained latent until the present

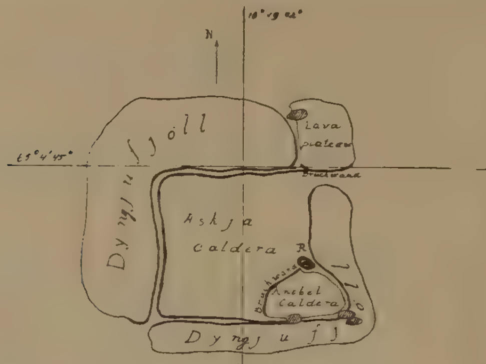

Fig. 3. Sketch of the Dyngjufjöll-Askja-Knebel-Complex. After H. RECK's topographical map [6], illustrating how the phenomenon of migration of younger eruption centres is repeated again and again.

Askja-caldera — Lava plateau
Askja-caldera — Knebel-caldera — Rudolf crater (R)
Knebel-caldera — S. E. craters (shaded)
Knebel-caldera — S. fumaroles „

day, whilst its previous history were unknown, (as is that of Ngorongoro, Askja-Knebel, and similar occurrences) then we should now search in vain, within the encompassing wall of the Kilauea-caldera, for the "central eruption channel" from which such a lava-floor emanated at one time.

From the foregoing it is clear that, in the course of time, the caldera of Kilauea went through a process of filling up identical with that of Vesuvius (17) (p. 1175—76). Far from being caused by downthrow, crumbling or explosions, it is this process which originated the successive terraces, black ledges, each younger one being broader and occupying a higher level than its predecessor, within the encompassure of the stable caldera wall. Why they will, inherently, be bordered, outwards *and* inwards, by *grosso modo* vertical walls, we have already explained (17).

Summarizing, we may conclude that the vertical wall encompassing the Kilauea-caldera is nothing but a top part of the cone pipe-wall of the old Kilauea-volcano, i.e. a product of primary origin. Further it is established that the successive terraces have been built up concomitantly with the narrowing-down process to which the magmatic conduit of Kilauea was subjected.

Still, there is no doubt that the phenomenon of subsidence and downthrow and temporary enlargement of the said conduit did take place at Kilauea at the close of various of its eruptive periods. And although of secondary

importance to the problem of caldera-*formation*, it is very likely that a misappreciation of the real nature of these phenomena and a false extrapolation furnished considerable support to the subsidence theory. Hence it seems necessary to identify that real nature.

Our analysis of the mechanism of volcanic cone-building (17) showed that cone-pipes in course of erection are being built up, generally speaking, under the influence of gravity, inter-friction and pressure of the extruding magmatic column on the accumulating ejectamenta; i.e. that of a gas-current laden with fragments and particles of eruptive and other rock-material, or of a column of more or less plastic rock material highly charged with magmatic vapours and gasses. At the same time we showed that the material constituting the conepipe-wall would, *eo ipso*, arrive in a state of *labile* equilibrium the moment the intensity of the pressure of the said magmatic column changed. Hence the occurrence of subsidence phenomena, falling in *of the orifice* of the cone-pipe and its subsequent, though often temporary, enlargement which is wont to follow immediately on periods of eruption, e.g., of strato-volcanoes (Vesuvius and others, [3]).

It is clear that these phenomena of subsidence, collapse and temporary enlargements of the cone-pipes of *lava-volcanoes (Schildvulkane)*, which are also wont to follow their periods of eruption (22) (23) (24), are identical in origin and nature with those of strato-volcanoes. In fact, under the influence of the cooling and solidifying effect of the upper surface and of the sides of the cone pipe-walls, a rising lava-column will tend to "cake on" (accretion, plastic lining, etc.) and so cause a shore or black ledge, of zonal structure. It is clear, however, that the viscosity (solidification) of such a shore or black ledge, will decrease with increasing depth; i.e. the cone pipe-wall will tend to liquefaction more and more with both, depth and vicinity to the incandescent magmatic rock increasing. Now, this peripherical part of such a zonal shore or black ledge will remain in a state of equilibrium, *grosso modo*, so long as the surface of the lava-column is at or near the level of its surface. On the retreat of the lava-column, however, such parts of the newly formed black ledge [1]) as have not reached a sufficient degree of solidification will tend to subside or collapse and in doing so may drag with them, to a greater or less degree, more distant parts of the black ledge or cone-pipe.

As a matter of fact the phenomenon described above was witnessed by PERRET at Halemaumau. He wrote (22):

"But the shore [2]) — undercut and plastic in its foundations — soon yielded to gravity as the support of the lava column was withdrawn, and settled......, whereupon the unsupported black ledge began to give way in a series of majestic downfalls...... Although on a smaller scale, the collapse of these previous formations was comparable to that of Vesuvius after the eruption of 1906. At the west and north sides the rock of the wall was

1) The "shore" by PERRET (see note 2).

2) Recently accretioned black ledge (see note 1).

under a powerful stress and detachment was accompanied by a sharp report as, with a crushing roar, the avalanche of broken rock descended in a cloud of stony dust...... At the east edge, on the contrary, detachment was gradual, the dislocated masses of rock sliding downward with a long-drawn roll of thunder......" (See also (24) a.o. p. 219, Conclusions.) (Compare also the descriptions of subsidence phenomena at the Knebel-caldera subsequent to its great eruption, by WATT, JOHNSTRUP and others [6] [12].)

It is clear that these phenomena do not affect the problem of caldera-*formation* in the least and that they concern, exclusively, that of an *enlargement*, often only temporary and relatively minimal at that, of an existing magmatic conduit *by means of subsidence of part of a newly formed or even adjacent black ledge along their planes of zonal accretion.*

It would seem that our study of the caldera-problem has led us to another important result. In fact, it has shown that the filling-up process of a caldera-space, often of gigantic dimensions, is accomplished by lateral accretions from a conduit and by depositions on its surroundings of the products of its overflow, in such a manner that an eruptive massive causally ensues in which zones of less and greater resistance will alternate, irregularly yet systematically. The divisional planes between successive layers of accretion and deposition will constitute, *grosso modo*, a system of planes of least resistance.

Moreover, it is clear that a tendency to a certain mode of orientation will be inherent in these alternating zones of less and greater resistance and that such orientation will largely correspond with the physical state of the rock-material concerned, during its deposition or accretion. Should the rock have been extremely liquid (Kilauea and, in general, the basaltic group of rocks), then it may be expected that the zones will be directed in a vertical and horizontal sense corresponding respectively with the orientation of the accretions and depositions. For more viscous material, these directions may be expected to vary accordingly.

Furthermore it is equally clear from the foregoing that certain directions of trend will be imposed genetically on these zones of less and greater resistance e.g. one which is parallel, generally speaking, to the wall of the eruption channel concerned.

Now, should our conclusions be right, then it may be reasonably expected that gravitative, magmatic, and other tensions will naturally tend for relief, along the directions indicated above, in the shape of faultplanes, subsidences, zones of disruption, fissures, eruption-points, fumaroles, etc. etc. Finally it should be kept in mind that such layers of accretion and deposition, may acquire any shape, size, and thickness, ranging from a thin wafer to a big mass, and that consequently their dividing planes, *which are of primary origin*, may be extremely close together (foliation, schistosity) or very far apart (banked condition etc.).

It follows that when in such and similar eruptive massives foliation,

divisional planes, fissures, faults, aligned eruption-points etc. are actually established, showing a pronounced and systematic orientation, such occurrence does not *eo ipso* justify the conclusion that these phenomena must have had a secondary origin i.e. were tectonically imposed after (or during) the consolidation of the massive, by the action of tangential or other forces.

As such massives may acquire huge dimensions — even supposing they cannot much surpass those we know already — it follows that our conclusions may fundamentally affect current principles on the origin and nature of the (tectonical) structure of eruptive masses and systems, *(Eruptiv Tektonik ;* (26)) as well as on the deductions based thereon. (Approximate areas of: Kilauea 14 km^2; Askja-Knebel 90 km^2; Ngorongoro 300 km^2; Ringgit 475 km^2 etc. etc.).

We will now put our conclusions to the test of facts which have been well established at Kilauea and in its neighbourhood.

An inter-comparison of several maps and sketches of Kilauea published since 1825 [13] [5] p. 463; (24) clearly demonstrates that zonal accretion and corresponding fissuring and faulting of the black ledge actually occurred and proceeded in a direction, *grosso modo*, parallel to the old Kilauea caldera-wall all round the still active remnant of its eruptive channel, Halemaumau.

Such orientation is proved by the trend of Lyman's ridge (e.f-g.h.), which indicates the inner border of DANA's black ledge of 1830 [13] (p. 85); by the system of contraction-planes with slight vertical displacements, which trend through the accreted black ledge of Kilauea, concentrical to the crater-wall of Halemaumau. We find our conclusions confirmed, not only by the mode of arrangement of outer-peripherically situated eruption channels round Kilauea as well as round Halemaumau (and also round Kilauea Iki, it would seem), but also by the arrangement of a series of smaller eruption points at or near the border of the black ledge of Kilauea (of 1840) (Fig. 1, *i, k, l, m*), and those within the outer wall of Halemaumau (*p, q, r, s.*). Last but not least, our conclusions are again supported by the directions of trend of the fault-planes going from W. by N. to E. in a wide semi-encompassing curve round the Kilauea-caldera, and comprising Kilauea-Iki and Keanakakoi.

By analogy with occurrences established in old Kilauea and round Halemaumau, it would seem justifiable to conclude that these systems of faults and fissures *round* Kilauea are either remnants of an older caldera-wall or else that they were originated in a black ledge of much larger dimensions, within which Kilauea and its smaller companions may have been the reduced eruption-channels of a larger volcano; just as Halemaumau is the reduced remnant of old Kilauea.

We should therefore not be surprised to discover that, in and round caldera-areas such as Dygnjufjöll, Santorin and others, tectonic and eruptive phenomena happen to be arranged in perfect harmony of orientation with the directory lines of such areas; *on the contrary we*

should expect such arrangement. We therefore emphasize that the establishment of their existence in such areas does not justify to invoke the fact of their occurrence as an irrefutable proof of the contention that such and similar calderas were *generated*, directly or indirectly, by these tectonical or eruptive phenomena (subsidence or explosion).

Such phenomena were *generated*, and they subsequently matured, at least to a large degree, in the body of the pre-existing caldera, as a *causal result* of the way in which it was filled up.

Finally we wish to add a few words to our previous argument. Firstly, it may be mentioned that Kilauea, like Vesuvius, furnished a striking example of the effects of re-fusion of part of its cone and a subsequent flow-back of the re-fused material (Fusion theory of HOCHSTETTER and others). Halemaumau went through such a process when, subsequent to its eruption of 1868, part of the cone all round its eruptive channel, having been re-fused, was capped over completely by a relatively very thin, semi-solid crust. Now, after the subsidence (back flow) of the liquid lava, its thin roofing collapsed; yet the caldera thus formed was, again, not bordered by vertical walls (compare (17) p. 1166).

As instances of blasting-out phenomena, with subsequent cone-pipe enlargement of minor importance (explosion theory (17) (p. 1166) we would mention those which occurred in Vesuvius in 1913 and in Krakatau in 1883.

It would not be difficult to furnish further evidence in support of our contentions, from the eruptive histories of other volcanoes, such as Santorin, Pelée and others. We preferred, however, to restrict our tests principally to well established facts in the history of Vesuvius and Kilauea because they represent two classical and beautiful types of their kind. Moreover, we know no other volcanoes which have been the object of such continuous, detailed, and scientific studies as have these two.

The chosen testing material was therefore of the first order and corresponding demands were therefore made on the arguments by which we had to vindicate our contentions.

The Hague, December 1927.

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Mathematics. — *On infinitesimal deformations of V_m in V_n.* By J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of November 26, 1927).

Let the points x^ν of a geodesic line in V_n be subjected to a transformation:

$$'x^\nu = x^\nu + \varepsilon v^\nu , \qquad (1)$$

where v^ν is a field of contravariant vectors defined along the line, and ε a small constant. Higher powers of ε may be neglected. Then we can deduce the conditions to which v^ν must conform in order that the transformed line may also be geodesic. A differential equation of the second order is found, which for $V_n = R_3$ is due to JACOBI, and in the general case to LEVI-CIVITA [1]).

An analogous question arises for a minimal-V_m in a V_n. This also leads to a differential equation of the second order, due for $V_n = R_3$, $m=2$, to SCHWARZ, and for general V_n and $m=2$ to CARTAN [2]).

We may now seek in general to find the equations expressing the change of the fundamental quantities of a V_m in V_n, when the points of this V_m are subjected to a displacement εv^ν. By fundamental quantities we understand the fundamental tensors and the different curvature quantities. After this we can easily find the differential equations for v^ν for the case that the displacement εv^ν does not change certain given properties of the V_m. It is only necessary to substitute the identities, characterizing this property, into the general equations.

In this paper we first deduce the conditions for a geodesic V_m and for a minimal-V_m, they are immediate generalisations of results found by LEVI-CIVITA and CARTAN; after this we deduce the equations for the bending [3]) of a V_m in V_n and find some interesting conclusions for the special case $V_n = R_n$. We conclude with the transformation of a V_{n-1} in V_n that leaves the principal directions of the second fundamental tensor invariant and with the equivoluminar transformation of a V_m in V_n.

§ 1. *The fundamental quantities of the V_m.*

We use two coordinate systems: $x^\nu, \lambda, \mu, \nu = a_1, \ldots, a_n$ in V_n and $y^c, a, b, c, d = 1, \ldots, m$ in V_m. According to a known property we can avoid the use of the coordinate system y. But it is useful for the present investigation, as we will accept that the deformation εv^ν takes it along

[1]) Sur l'écart géodésique, Math. Ann. **97** (26), 291—320.

[2]) Sur l'écart géodésique et quelques notions connexes, Rend. Acc. Lincei (6a) **5** (27) 609—613.

[3]) Under bending we understand a flexion without tearing or stretching.

with it. Hence the quantities of the V_n carry Greek, those of the V_m Latin indices. Under transformations of the x the components of the quantities of the V_m do not change, being fixed with respect to the y. The same holds for the components of the quantities of the V_n under a transformation of the y. Furthermore we have quantities with both Greek and Latin indices, their components being changed as the components of a quantity of V_m in as much as the Greek indices are concerned, and as those of a quantity of V_n in as much as the Latin indices are concerned [1]).

The most important of these quantities is:

$$B_a^\nu = \frac{\partial x^\nu}{\partial x^a} \quad \ldots \ldots \ldots \ldots \quad (2)$$

It can be shown very easily indeed that the B_a^ν behave like the components of a contravariant vector of the V_n under transformations of the x; and like components of a covariant vector under transformations of the y. With the aid of B_a^ν we deduce from the fundamental tensor $g_{\lambda\mu}$ of the V_n the fundamental tensor of the V_m:

$$g'_{ab} = B_{ab}^{\lambda\mu}\, g_{\lambda\mu} \quad ; \quad (B_{ab}^{\lambda\mu} = B_a^\lambda\, B_b^\mu) \quad \ldots \ldots \quad (3$$

and with the aid of g'_{ab} we form the quantity:

$$B_\lambda^c = g'^{cb}\, B_b^\mu\, g_{\lambda\mu} \quad \ldots \ldots \ldots \ldots \quad (4)$$

It follows from (2) and (4) that

$$B_\mu^c\, B_a^\mu = B_a^c\,, \quad \ldots \ldots \ldots \ldots \quad (5)$$

here we denote with B_a^c a set of m^2 numbers with value 1 when $a = c$, and 0 when $a \neq c$.

The quantities B_a^ν and B_λ^c allow us to define a unique correspondence between the quantities of the V_m and some of the quantites of the V_n and vice versa. We only need to show it for vectors. If

$$\overline{v}^\nu = B_b^\nu\, v^b \quad \ldots \ldots \ldots \ldots \quad (6^a)$$

corresponds to v^c, or, in covariant components:

$$\overline{v}_\lambda = g_{\lambda\nu}\, B_b^\nu\, g'^{bc}\, v_c = B_\lambda^c\, v_c\,, \quad \ldots \ldots \ldots \quad (6^b)$$

we have, on account of (5):

$$B_\mu^c\, \overline{v}^\mu = v^c \quad \ldots \ldots \ldots \ldots \quad (7^a)$$

$$B_a^\mu\, \overline{v}_\mu = v_a \quad \ldots \ldots \ldots \ldots \quad (7^b)$$

Hence it is equally possible to deduce v from $\overline{v}$ as $\overline{v}$ from v. We use this property to write v^ν and v_λ in stead of $\overline{v}^\nu$ and $\overline{v}_\lambda$; we therefore consider *them as another kind of components* of the vector v. We have reached

[1]) Such quantities are already introduced, for the discussion of V_m in V_n, by E. BOMPIANI, Studi sugli spazi curvi, Atti Veneto **80** (20/21) 1113—1145, and more systematically by B. L. VAN DER WAERDEN, Differentialkovarianten von V_m in V_n, Abh. Math. Sem. Hamburg **5** (27) 153—160.

by proceeding in this way, that all vectors, and therefore all quantities, of the V_m can be considered as quantities of the V_n. On this property depends the mentioned possibility to discard the y totally.

We say that a vector v^ν, defined with respect to the V_n, *lies in the* V_m when $v^\nu = B^\nu_\lambda v^\lambda$. The geometrical meaning is clear: the direction of v^ν is tangent to V_m. It is obvious that a vector, not lying within the V_m, has no components with Latin indices. It is of course possible to form $B^c_\mu v^\mu$, but these are the components of the projection of v on V_m, not of v itself. In the same way we see that a quantity $P_{\lambda\mu\nu}$ has only then components $P_{ab\nu}$, if it "lies in the V_m with the indices λ and μ", that is to say, if $B^{\alpha\beta}_{\lambda\mu} P_{\alpha\beta\nu} = P_{\lambda\mu\nu}$.

A vectorfield v_λ, defined in V_n, has a covariant derivative

$$\nabla_\mu v_\lambda = \frac{\partial v_\lambda}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} v_\nu \quad ; \quad \Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \lambda\,\mu \\ \nu \end{matrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\left\{ \begin{smallmatrix} \lambda\mu \\ \nu \end{smallmatrix} \right\}$ are the CHRISTOFFEL symbols belonging to the $g_{\lambda\mu}$. In the same way a vector field w_a, defined in V_m, has a covariant derivative, whose components with Latin indices are

$$\nabla'_b w_a = \frac{\partial w^a}{\partial y^b} - \Gamma'^c_{ab} w_c \quad ; \quad \Gamma'^c_{ab} = \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\}' . \quad . \quad . \quad . \quad . \quad (9)$$

and with Greek indices:

$$\nabla'_\mu w_\lambda = B^{ba}_{\lambda\mu} \frac{\partial w_a}{\partial y_b} - B^{ba}_{\mu\lambda} \Gamma'^c_{ab} w_c \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$\left\{ \begin{smallmatrix} ab \\ c \end{smallmatrix} \right\}'$ are the CHRISTOFFEL symbols belonging to the g'_{ab}. For a vector field u_λ of V_n, defined on V_m, the expression $\nabla_\mu u_\lambda$ has no meaning. But the expression

$$B^\alpha_\mu \nabla_\alpha u_\lambda = B^a_\mu B^\alpha_a \nabla_\alpha u_\lambda = B^a_\mu \frac{\partial u_\lambda}{\partial y^a} - B^\alpha_\mu \Gamma^\nu_{\lambda\alpha} u_\nu \quad . \quad . \quad . \quad (11)$$

has certainly a meaning, and represents another kind of covariant derivative. It can easily be proved, that for the case of u_λ lying in V_m:

$$\nabla'_b u_a = B^{\mu\lambda}_{ba} \nabla_\mu u_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the same way we can build different kinds of derivatives for quantities of higher order of the V_n, defined on the V_m. One of the most frequently occurring quantities is $B^{\beta\alpha}_{\omega\mu} \nabla_\beta v_{\alpha\lambda}$, where $v_{\mu\lambda}$ is a field with index μ within V_m. For the *baλ*-component of this derivative we easily find

$$B^{\beta\alpha}_{ba} \nabla_\beta v_{\alpha\lambda} = \frac{\partial v_{a\lambda}}{\partial y_b} - \Gamma'^c_{ab} v_{c\lambda} - B^\mu_b \Gamma^\nu_{\lambda\mu} v_{a\nu} \text{ [1]} \quad . \quad . \quad . \quad . \quad (13)$$

[1] The factors B could be avoided in most cases by the introduction of new differentiation symbols for the different derivatives, e.g. $\overset{1}{\nabla}$, $\overset{2}{\nabla}$. If however in this way we want to come to a systematic notation applyable to all cases, the sign ∇ must indicate in which way the factors B affect the indices. This makes the notation less clear and more complicated than the notation used here as well as in Chapter III and IV of "Der Ricci Kalkül", SPRINGER 1924.

Together with the fundamental tensor the following fundamental quantities are the most important [1]).

1st *The curvature affinor.*

$$H_{\lambda\mu}^{\cdot\cdot\,\nu} = B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This quantity lies with its first two indices in the V_m. Hence it has also components with two Latin indices:

$$H_{ab}^{\cdot\cdot\,\nu} = B_{ab}^{\lambda\mu} H_{\lambda\mu}^{\cdot\cdot\,\nu} = B_{ab}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The vanishing of $H_{ab}^{\cdot\cdot\,\nu}$ is necessary and sufficient for V_m being geodesic. For $m = n - 1$ $H_{ab}^{\cdot\cdot\,\nu}$ passes into $-h_{ab} n^\nu$, where h_{ab} is the second fundamental tensor and n^ν the unit vector normal to V_{n-1}.

2nd *The mean curvature vector.*

$$D^\nu = \frac{1}{m} g'^{ab} H_{ab}^{\cdot\cdot\,\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Its vanishing is necessary and sufficient for V_m being a minimal manifold. For $m = n - 1$ we have $-hn^\nu = -h_{\cdot a}^{a} n^\nu$ in stead of mD^ν.

§ 2. *The fundamental equations.*

Under a deformation εv^ν these quantities are changed in the following way:

I. $\delta g'_{\lambda\mu} = 2\,\varepsilon\, B_{(\lambda}^{\alpha} C_{\mu)}^{\beta} \nabla_\alpha v_\beta$ II. $dg'_{ab} = 2\,\varepsilon\, B_{(ab)}^{\alpha\beta} \nabla_\alpha v_\beta$

$$\text{III.}\quad \delta H_{\lambda\mu}^{\cdot\cdot\,\nu} = 2\,\varepsilon\, H_{\cdot(\lambda}^{\alpha\cdot\,\nu} C_{\mu)}^{\beta} \nabla_\alpha v_\beta - 2\,\varepsilon\, H_{\cdot(\lambda}^{\beta\cdot\,\nu} B_{\mu)}^{\alpha} \nabla_\alpha v_\beta - \varepsilon\, H_{\lambda\mu}^{\cdot\cdot\,\beta} g'^{\nu\alpha} \nabla_\alpha v^\beta -$$
$$- \varepsilon\, B_{\lambda\mu}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} v^\gamma + \varepsilon\, C_\delta^\nu B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\gamma \nabla_\gamma v^\delta \,.$$

$$\text{IV.}\quad dH_{ab}^{\cdot\cdot\,\nu} = - \varepsilon\, H_{ab}^{\cdot\cdot\,\beta} g'^{\nu\alpha} \nabla_\alpha v_\beta - \varepsilon\, B_{ab}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} v^\gamma +$$
$$+ \varepsilon\, C_\alpha^\nu B_{ab}^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \varepsilon\, H_{ab}^{\cdot\cdot\,\alpha} \Gamma_{\alpha\beta}^{\nu} v^\beta \,.$$

$$\text{V.}\quad \delta D^\nu = - \varepsilon\, D^\beta g'^{\nu\alpha} \nabla_\alpha v_\beta - \frac{1}{m} \varepsilon\, C_\delta^\nu g'^{\alpha\beta} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} v^\gamma +$$
$$+ \frac{1}{m} \varepsilon\, C_\alpha^\nu g'^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \frac{1}{m} \varepsilon\, H^{\alpha\beta\nu} \nabla_\alpha v_\beta \,.$$

$$\text{VI.}\quad dK'_{abcd} = - 4\,\varepsilon\, B_{[a\,[c}^{\alpha\beta} H_{b]\,d]}^{\cdot\cdot\,\delta} K_{\gamma\alpha\beta\delta} v^\gamma + 4\,\varepsilon\, H_{[a\,[c}^{\cdot\cdot\,\alpha} B_{b]d]}^{\beta\delta} \nabla_\beta B_\delta^\gamma \nabla_\gamma v_\alpha$$
$$+ \varepsilon\, B_{abcd}^{\omega\mu\lambda\nu} \{K_{\alpha\mu\lambda\nu} \nabla_\omega v^\alpha + K_{\omega\alpha\lambda\nu} \nabla_\mu v^\alpha + K_{\omega\mu\alpha\nu} \nabla_\lambda v^\alpha + K_{\omega\mu\lambda\alpha} \nabla_\nu v^\alpha \}$$
$$+ \varepsilon\, v^\varepsilon B_{abcd}^{\alpha\beta\gamma\delta} \nabla_\varepsilon K_{\alpha\beta\gamma\delta}.$$

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[1]) Compare e.g. Chapter V of "Der Ricci Kalkül" (further on referred to as R.K.).

$$\text{VII.}\ dK'_{bc} = \varepsilon\,(-B_c^{\beta}\,H_b^{\cdot\,\alpha\delta} + m\,B_{bc}^{\alpha\beta}\,D^{\delta} - B_b^{\alpha}\,H_c^{\cdot\,\beta\delta} + g'^{\alpha\beta}\,H_{bc}^{\cdot\cdot\,\delta})\,v^{\gamma}\,K_{\gamma\alpha\beta\delta} +$$

$$+ 4\,\varepsilon\,g'^{ad}\,H_{[a[c}^{\cdot\cdot\,\alpha}\,B_{b]d]}^{\beta\delta}\,\nabla_{\beta}\,B_{\delta}^{\chi}\,\nabla_{\chi}\,v_{\alpha}$$

$$+ 2\,\varepsilon\,B_{bc}^{\mu\lambda}\,g'^{\omega\nu}\,K_{\alpha(\lambda\mu)\nu}\,\nabla_{\omega}\,v^{\alpha} + 2\,\varepsilon\,B_{bc}^{\mu\lambda}\,g'^{\omega\nu}\,K_{\alpha\omega\nu(\lambda}\,\nabla_{\mu)}\,v^{\alpha} -$$

$$- 2\,\varepsilon\,K'_{abcd}\,g'^{a\lambda}\,g'^{d\mu}\,\nabla_{(\mu}\,v_{\lambda)} + \varepsilon\,v^{\varepsilon}\,g'^{\alpha\delta}\,B_{bc}^{\beta\gamma}\,\nabla_{\varepsilon}\,K_{\alpha\beta\gamma\delta}.$$

$$\text{VIII.}\ dK' = -2\,\varepsilon\,H^{\beta\alpha\delta}\,v^{\gamma}\,K_{\gamma\alpha\beta\delta} + 2\,m\,\varepsilon\,g'^{\alpha\beta}\,D^{\delta}\,v^{\gamma}\,K_{\gamma\alpha\beta\delta} +$$

$$+ 2\,\varepsilon\,H^{\beta\gamma\alpha}\,\nabla_{\beta}\,B_{\gamma}^{\delta}\,\nabla_{\delta}\,v_{\alpha} - 2\,m\,\varepsilon\,D^{\alpha}\,g'^{\beta\gamma}\,\nabla_{\beta}\,B_{\gamma}^{\delta}\,\nabla_{\delta}\,v_{\alpha}$$

$$+ 4\,\varepsilon\,K_{\alpha\mu\lambda\nu}\,g'^{\omega\nu}\,g'^{\mu\lambda}\,\nabla_{\omega}\,v^{\alpha} - 4\,\varepsilon\,K'^{\alpha\beta}\,\nabla_{\alpha}\,v_{\beta} + \varepsilon\,v^{\varepsilon}\,g'^{\alpha\delta}\,g'^{\beta\gamma}\,\nabla_{\varepsilon}\,K_{\alpha\beta\gamma\delta}.$$

We obtain (I) starting from (1) and (2). From (I) equations (II) are deduced. For $m = n$ (II) passes into the well known equation for the variation of the fundamental tensor of the V_n under an infinitesimal transformation [1]). We obtain (III) and (IV) from (I) and (14); and (V) from (11). (VI—VIII) are deduced from (IV) and GAUSS' equation. For $m = n$ the quantities $H_{\mu\lambda}^{\cdot\cdot\,\nu}$ and D^{ν} vanish, and (VI) passes into the equation expressing the change of the curvature quantity under an infinitesimal transformation [2]). For $m = n - 1$ we have in stead of III, IV and V:

$$\text{III}'.\ \delta h_{\lambda\mu} = +2\,\varepsilon\,h^{\alpha}_{\cdot\,(\mu}C_{\lambda)}^{\beta}\,\nabla_{\alpha}\,v_{\beta} - 2\,\varepsilon\,h^{\beta}_{\cdot\,(\mu}B_{\lambda)}^{\alpha}\,\nabla_{\alpha}\,v_{\beta} +$$

$$+ \varepsilon\,B_{\lambda\mu}^{\alpha\beta}\,K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\,n_{\delta}\,v^{\gamma} - \varepsilon\,n_{\alpha}\,B_{\lambda\mu}^{\beta\gamma}\,\nabla_{\beta}\,B_{\gamma}^{\delta}\,\nabla_{\delta}\,v^{\alpha}.$$

$$\text{IV}'.\ dh_{ab} = \varepsilon\,B_{ab}^{\alpha\beta}\,K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\,n_{\delta}\,v^{\gamma} - \varepsilon\,n_{\alpha}\,B_{ab}^{\beta\delta}\,\nabla_{\beta}\,B_{\delta}^{\gamma}\,\nabla_{\gamma}\,v^{\alpha}.$$

$$\text{V}'.\ dh_a^{\cdot\,a} = \varepsilon\,K_{\alpha\beta}\,n^{\alpha}\,v^{\beta} - \varepsilon\,n_{\alpha}\,g'^{\beta\gamma}\,\nabla_{\beta}\,B_{\gamma}^{\delta}\,\nabla_{\delta}\,v^{\alpha} - 2\,\varepsilon\,h^{\alpha\beta}\,\nabla_{\alpha}\,v_{\beta}.$$

If we decompose in this case v^{ν} into a component w^{ν} in the V_{n-1} and another, ψn^{ν}, normal to the V_{n-1} (n^{ν} being unit vector) we find

$$B_{\mu}^{\alpha}\,\nabla_{\alpha}\,v_{\lambda} = \nabla'_{\mu}\,w_{\lambda} + \psi\,h_{\mu\lambda} - h_{\mu}^{\alpha}\,w_{\alpha}\,n_{\lambda} + n_{\lambda}\,\nabla'_{\mu}\psi \quad . \quad . \quad (17)$$

$$n^{\alpha}\,B_{\omega\mu}^{\beta\gamma}\,\nabla_{\beta}\,B_{\gamma}^{\delta}\,\nabla_{\delta}\,v_{\alpha} = -\,h_{\omega}^{\cdot\,\alpha}\,\nabla'_{\mu}\,w_{\alpha} - \nabla'_{\omega}\,h_{\mu}^{\alpha}\,w_{\alpha} - \psi\,h_{\omega}^{\cdot\,\alpha}\,h_{\mu\alpha} + \nabla'_{\omega}\nabla'_{\mu}\,\psi. \quad (18)$$

The equation of KILLING $\nabla_{(\mu}\,v_{\lambda)} = 0$ [3]) is characteristic for the rigid motions in V_n. It can indeed be shown without difficulty that in this case all diffferentials vanish.

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[1]) R.K., p. 209.
[2]) R.K., p. 208.
[3]) R.K., p. 212.

§ 3. *Geodesic V_m.*

Necessary and sufficient condition that a geodesic V_m remains geodesic under a deformation εv^ν, is, after (IV), that

$$C^\nu_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} C^\nu_\delta K_{\gamma\alpha\beta}{}^{\cdot\cdot\cdot\delta} v^\gamma = 0 \quad . \quad . \quad . \quad . \quad (19)$$

For $m = n - 1$ this equation passes into

$$n_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} K_{\alpha\gamma\beta\delta} v^\gamma n^\delta = 0 \quad . \quad . \quad . \quad . \quad (20)$$

and for $m = 1$ into:

$$\frac{\delta^2}{ds^2} v^\delta - i^\alpha i^\beta K_{\gamma\alpha\beta}{}^{\cdot\cdot\cdot\delta} v^\gamma = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the equation of LEVI-CIVITA.

§ 4. *Minimal-V_m.*

Necessary and sufficient condition that the minimal property is not changed is

$$C^\nu_\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - C^\nu_\delta g'^{\alpha\beta} K_{\gamma\alpha\beta}{}^{\cdot\cdot\cdot\delta} v^\gamma - H^{\alpha\beta\nu} \nabla_\alpha v_\beta = 0. \quad . \quad (22)$$

For $m = 2$ and $v^\nu \perp V_2$ this equation is equivalent to CARTAN's equation.

For $m = n - 1$ equation (22) passes into

$$n^\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v_\alpha - K_{\alpha\beta} n^\alpha v^\beta + h^{\alpha\beta} \nabla_\alpha v_\beta = 0 \quad . \quad . \quad . \quad (23)$$

If we take $v^\nu = \psi n^\nu$ and the unit vector $n^\nu \perp V_{n-1}$, we get

$$\nabla'^a \nabla'_a \psi - \psi K_{\alpha\beta} n^\alpha n^\beta + \psi h^{\alpha\beta} h_{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

and, if in this case $V_n = R_3$, $m = 2$, we obtain the equation of SCHWARZ

$$\nabla'^a \nabla'_a \psi - 2 K'_0 \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in wich K'_0 is the curvature of the V_2.

§ 5. *Bending.*

A V_m is bended when its metric is not changed under the deformation. Hence the necessary and sufficient condition is $dg'^{ab} = 0$, or, with respect to (II)

$$B^{\alpha\beta}_{ab} \nabla_{(\alpha} v_{\beta)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For $m = n - 1$ we get from (17)

$$\nabla'_{(a} w_{b)} = - \psi h_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If $V_n = R_n$ and the rank of h_{ab} is larger than 1, we can obtain from this differential equation an equation of the second order with $\nabla_{[a} w_{b]}$ as dependent variable, which does no longer contain the function ψ. If we write $\nabla'_{[a} w_{b]} = f_{ab}$, the integrability conditions of (27) are:

$$\tfrac{1}{2} K'_{cab}{}^{\cdot\cdot\cdot d} w_d - \nabla'_{[c} f_{a]b} = - (\nabla'_{[c} \psi) h_{a]b}, \quad . \quad . \quad . \quad . \quad (28)$$

For the deduction of the right member we have used CODAZZI's equation

$$\nabla'_{[c} h_{a]b} = 0 \quad \ldots\ldots\ldots\ldots \quad (29)$$

From (28) we obtain, making use of GAUSS' equation:

$$h_{b[c} h_{a]}^{\cdot d} w_d - \nabla'_{[c} f_{a]b} = -(\nabla'_{[c} \psi) h_{a]b} \quad \ldots\ldots \quad (30)$$

Transvection with h^{ab} gives

$$p_c^{\cdot a} h_a^{\cdot d} w_d + h^{ab} \nabla'_a f_{cb} = p_c^{\cdot a} \nabla'_a \psi \quad \ldots\ldots \quad (31)$$

in which we have used the abbreviation

$$h^{ab} h_{cb} - h^{db} h_{db} A_c^a = p_c^{\cdot a} \quad \ldots\ldots\ldots\ldots \quad (32)$$

The rank of p_{ab} is always n when the rank of h_{ab} is not equal to 1. Hence there exists an inverse tensor P_{ab} of p_{ab} and the introduction of P_{ab} into equation (31) gives the simpler formula

$$h_c^{\cdot a} w_a + P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = \nabla'_c \psi \quad \ldots\ldots\ldots \quad (33)$$

from which we obtain by differentiation and alternation

$$\boxed{h_{[c}^{\cdot a} f_{d]a} = \nabla'_{[c} u_{d]}} \quad \ldots\ldots\ldots\ldots \quad (A)$$

in which we have used the abbreviation:

$$P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = u_c \quad \ldots\ldots\ldots\ldots \quad (34)$$

If (33) is substituted into (17), we get, making use of (27) and (34)

$$B_\mu^\alpha \nabla_\alpha v_\lambda = f_{\mu\lambda} + u_\mu n_\lambda \quad \ldots\ldots\ldots \quad (35)$$

The integrability conditions of this equation are, so far as they are not a consequence of (A):

$$\boxed{\nabla'_a f_{bc} = 2\, h_{a[b} u_{c]}} \quad \ldots\ldots\ldots\ldots \quad (B)$$

It can easily be shown that the integrability conditions of (A) are a consequence of (B). Those of (B) are

$$\boxed{h_{[a[c} h_{b]}^{\cdot \alpha} f_{d]\alpha} = h_{[a[c} \nabla'_{b]} u_{d]}} \quad \ldots\ldots\ldots \quad (C)$$

and the integrability conditions of this equation are a consequence of (B). Hence the system (A), (B), (C) is complete [1]). If we compute for this case the right side of equation (VI), we see that (C) expresses the fact that $dK'_{abcd} = 0$.

Now we will first investigate under which conditions the bending is improper, that is to say, is only a pure motion of the R_n. Necessary and sufficient condition for this is that we can find a vector field in the R_n being equal to v_λ in every point of the V_m and being choosen in other points in such a way, that

$$\nabla_\mu v_\lambda = \nabla_{[\mu} v_{\lambda]} = F_{\mu\lambda} \quad \ldots\ldots\ldots\ldots \quad (36)$$

1) The equations (A), (B) and (C) are equivalent with a system deduced by SBRANA: Sulla deformazione infinitesima delle ipersuperficie, Ann. di Mat. (3) **15** ('08) 329—348.

is a constant bivector in R_n; that is to say that there exists in V_m a vector field p_λ, such that the bivector

$$F_{\mu\lambda} = f_{\mu\lambda} + 2 p_{[\mu} i_{\lambda]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

is constant in R_n. This is however then and only then the case, when the system

$$\boxed{h_c^{\cdot a} f_{da} = \nabla_c p_d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_0)$$

$$\boxed{\nabla'_a f_{bc} = 2 h_{a[b} p_{c]}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_0)$$

admits a solution. It can be shown without difficulty that (A_0) and (B_0) also form a complete system, the equation corresponding tot (C) being here a consequence of (A_0). If a solution of (A_0, B_0) is found, we have for the corresponding motion

$$B_\mu^\alpha \nabla_\alpha v_\lambda = f_{\mu\lambda} + p_\mu i_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Hence a solution of (A, B, C) is then and only then not a proper bending, if this solution also satisfies (A_0) for $u = p$.

Now the following theorem holds and can be proved easily by writing out the components with respect to the principal directions of h_{ab}:

Given the equation $h_{[a[c} k_{b]d]} = 0$, in which h_{ab} is real and symmetrical and k_{ab} arbitrary. Then if h_{ab} has the rank 2, k_{ab} lies totally in the R_2 of h_{ab}, and if h_{ab} has a rank > 2, k_{ab} vanishes.

Hence we deduce from (C) that a V_m in R_n admits only then proper bendings, if the rank of h_{ab} is 2 or less, a well-known property, first published by KILLING [1]). If the rank of h_{ab} is 2, we have the only case that the ∞^{n-3} directions of h_{ab} form, at each point, a *plane* R_{n-3} lying totally in the V_{n-1} and with the same tangent-R_{n-1} at each point. This was proved bij BOMPIANI [2]). Hence the V_{n-1} is thus built up by ∞^2 of such R_{n-3}. According to (B) $f_{\lambda\mu}$ is constant in each of these R_{n-3}. If the above mentioned theorem is applied to (C), we find that $h_b^{\cdot\alpha} f_{d\alpha} - \nabla'_b n_d$ lies totally in the R_2 of $h_{\lambda\mu}$; and this shows that also u_a in each of the R_{n-3} of V_{n-1} is constant. We have besides, from (IV') and (35):

$$d\, h_{ab} = - h_a^{\cdot\alpha} f_{b\alpha} + \nabla'_a u_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Hence if y^1 and y^2 are chosen in such a way that the R_{n-3} become

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1) Die nichteuklidischen Raumformen in analytischer Behandlung, Leipz., 1885, p. 236 a.f.

2) Forma geometrica delle condizione per la deformabilità delle ipersuperficie, Rend. Acc. Lincei (5) 23. 1 (14) 126—131. The first part is an immediate consequence of CODAZZI's equation, if written in orthogonal components with respect to the principal direction of h_{ab}, the second part follows from the geometrical meaning of $B_\mu^\alpha \nabla_\alpha n_\lambda$. Comp. STRUIK, Grundzüge der mehrdimensionalen Differentialgeometrie, SPRINGER 1922, p. 140. CARTAN, La déformation des hypersurfaces dans l'espace euclidéen réel à n dimensions, Bull. Soc. Math. de France 44 (16) 65—99, has a complete classification of all possible cases where a V_{n-1} is bended in a R_n.

the intersections of the systems of V_{n-2} belonging to y^1 and y^2, we see from this last equation that dh_{ab} always vanishes except for $n \leqq 2$ and $b \leqq 2$ simultaniously. Hence the plane R_{n-3} remain plane under bending. [1]).

Starting with a definite solution v_λ of (A, B, C) we can always obtain that $f_{\mu\lambda}$ and u_λ vanish at some point P, without essential change of the solution. We have only to determine the corresponding proper motion v'^λ, giving at P the same value for $f_{\mu\lambda}$ and u_λ. Then $v_\lambda - v'_\lambda$ is the desired solution not differing essentially from v_λ. This will be called the *reduction* of the bending with respect to P.

Let us give besides a short treatment of the case $m=2$, in which case $f_{\mu\lambda}$ passes into $\varphi I_{\mu\lambda}$; $I_{\mu\lambda}$ being the unit bivector of the V_2. The function φ is WEINGARTEN's "Verschiebungsfunktion" [2]). Then equations (A, B, C) become:

$$\boxed{\varphi\, h_{[c}^{\cdot\,a} I_{d]a} = \nabla'_{[c}\, u_{d]}} \qquad (A_1)$$

$$\boxed{\nabla'_a \varphi = {}^1/_2\, h_{ab}\, I^{bc}\, u_c} \qquad (B_1)$$

$$\boxed{0 = h_{[a\,[c}\, \nabla'_{b]}\, u_{d]}} \qquad (C_1)$$

From (A_1) and (B_1) we obtain:

$$\nabla'_a\, H^{ab} \nabla'_b\, \varphi = -{}^1/_8\, \varphi\, h^{a}_{\cdot\, a} \qquad (40)$$

in which H^{ab} is reciprocal to h_{ab}.

This equation comes in stead of (A) and is equivalent to the characteristic equation of WEINGARTEN [3]). It can easily be shown that every value of φ deduced from a solution of the characteristic equation satisfies (C_1) identically.

A remarkable case of proper bending is that in which the $(n-1)$-direction of each element of the V_{n-1} remains unchanged. The necessary and sufficient condition for this is that not only dg'_{ab}, but also $\delta g'_{\lambda\mu}$ vanishes. This occurs then and only then if, as we see from (I) and (II) $B_\mu^\alpha \nabla_\alpha v_\lambda$ lies totally in V_{n-1} and is at the same time alternating. Then the equations (A, B, C) pass into

$$\boxed{h_{[c}^{\cdot\,a} f_{d]a} = 0} \qquad (A_2)$$

$$\boxed{\nabla'_a f_{bc} = 0} \qquad (B_2)$$

$$\boxed{h_{[a\,[c}\, h_{b]}^{\cdot\,\alpha} f_{d]\alpha} = 0} \qquad (C_2)$$

[1]) BOMPIANI, Forma geometrica delle condizioni per la deformabilità delle superficie, Rend. Linc. 33 (14) 126—131.

[2]) Comp. e.g. BIANCHI-LUKAT, Vorlesungen über Differentialgeometrie, Leipzig, 1899, p. 289 a.f.

[3]) E.g. BIANCHI-LUKAT, l.c. p. 292, equation (7*).

It follows from (C_2) that, except for a scalar factor, f_{ab} is equal to the unit bivector in the R_2 of h_{ab}. It follows from (B_2), that this scalar factor is a constant and that the R_2 of h_{ab} is geodesically parallel (in V_{n-1}) at all points of V_{n-1}. If (A_2) is written out in orthogonal components with respect to the principal axes of h_{ab}, it appears that this equation is then and only then satisfied if the V_{n-1} is minimal. If we reduce the bending with respect to some point P, we see that there is essentially only one solution. Hence we have obtained the theorem, obtained by DARBOUX for the case of a V_2 in R_3 [1]):

Necessary and sufficient condition that a V_{n-1} in R_n, for which h_{ab} has the rank 2, can be subjected to a proper infinitesimal bending, with preservation of the (n—1)-direction of each element, is, that the V_{n-1} be a minimal-V_{n-1} and that the R_2 of h_{ab} be geodesically parallel in V_{n-1} at all points of V_{n-1}. If one such a bending is given, then any other can be obtained from it by the adjunction of a proper motion.

§ 7. *Infinitesimal deformations normal to V_{n-1} that keep the principal directions of h_{ab} invariant.*

Suppose $v^\nu \perp V_{n-1}$. Then we have from (IV′) and (18):

$$d\,h_{ab} = \varepsilon\,\psi\,B_{ab}^{\alpha\beta}\,K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\,n_\delta\,n^\gamma + \varepsilon\,\psi\,h_a^{\cdot\,c}\,h_{bc} - \varepsilon\nabla'_a\nabla'_b\,\psi \quad . \quad . \quad (41)$$

The tensor $h_a^{\cdot\,c}\,h_{bc}$ has the same principal directions as h_{ab}. Hence the necessary and sufficient condition that the principal directions of h_{ab} remain invariant, is:

$$\underset{a}{i^\alpha}\,\underset{b}{i^\beta}\,(\nabla'_\alpha\nabla'_\beta\,\psi - \psi\,K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\,n_\delta\,n^\gamma) = 0; \quad a, b = 1, \ldots, n-1, \; a \neq b, \quad (42)$$

in which the i are unit vectors in the principal directions of h_{ab}. Such a transformation exists when we pass from one of the V_{n-1} of an n-uple orthogonal system to a neighbouring V_{n-1}. It can be shown indeed, that in this case one of the conditions for the existence of such a system is given by equation (42) [2]).

§ 8. *Infinitesimal transformations that keep invariant the the m-dimensional volume.*

The volume of the parallepiped with sides $dy^1, dy^2, \ldots, dy^m$ is, according to a well-known formula:

$$d\tau = dy^1 \ldots dy^m \sqrt{g'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

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[1]) Leçons sur la théorie générale des surfaces, Paris, 1914, I. p. 383.

[2]) For literature compare SCHOUTEN and STRUIK, On n-uple orthogonal systems of V_{n-1} in V_n, These Proceedings 22, (1919), p. 594–605, 680—695.

A transformation is equivoluminar, when $d\tau$ remains invariant, or in consequence of (43) when $d\sqrt{g'}=0$. But

$$d\sqrt{g'}={}^1/_2\, g'^{-1/2}\, dg'={}^1/_2 \sqrt{g'}\, g'^{ab}\, dg'_{ab}=\varepsilon \sqrt{g'}g'^{\mu\lambda}\nabla_\mu v_\lambda \quad . \quad (44)$$

This formula passes for $m=n-1$ into

$$d\sqrt{g'}=\varepsilon\, \psi \sqrt{g'}\,(\nabla'^a\, w_a+\psi\, h^{a}_{.a}) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

on account of (17). This shows that an infinitesimal transformation of a V_{n-1} in V_n perpendicular to this V_{n-1} is then and only then equivoluminar if the V_{n-1} is minimal. This theorem is due to Bompiani.[1] [2]).

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[1]) Studi sugli spazi curvi, Atti del R.I. Veneto 80. 2 (20/21) 1113—1145, p. 1141.

[2]) In a recent dissertation at the Massachusetts Institute of Technology, with title "Infinitesimal Deformation of Surfaces in Riemannian Space", W. F. Cheney investigates the bending of V_2 in V_n, especially for the cases $n=3$, $n=4$. In these cases he comes to equations, which for the case $V_n=R_n$ correspond to the equations (A_1, B_1) of our paper. An abstract of this dissertation is published in "Abstracts of Publications of the Massachusetts Institute of Technology", Vol. I (1928).

Physiology. — *Experimental contributions to the knowledge concerning the segmental innervation of the abdominal muscles in the dog.* (1st Communication.) *General statement of problem and I. The M. Rectus Abdominis* [1]). By Prof. G. VAN RIJNBERK and Miss L. KAISER.

(Communicated at the meeting of January 28, 1928).

GENERAL STATEMENT OF PROBLEM.

The muscles of the abdominal wall hold a somewhat special position among the muscles of the trunk in regard of problems of segmental anatomy. The structure of those muscles reminds us of the original myomery by the presence in many places of obvious rudiments of myosepta; besides the innervation being plainly metamerical in type. In the following communications we will give some contributions to a more exact knowledge of the actual facts concerning the segmental structure and radicular innervation (rhizomery) of those muscles and also of the mutual relation of those two.

I. RECTUS ABDOMINIS MUSCLE.

ANATOMICAL INTRODUCTION.

Gross Anatomy.

The M. Rectus Abdominis consists in dog in one long narrow layer of muscle that is inserted craniad with a relatively broad head to the 1st—5th ribs, a fascia intervening. Caudad the insertion consists of a narrow strip fixed to the pubal symphysis. The fibres take a parallel course, in cranial-caudal direction. The larger part of its course is intersected by fibrous partitions plainly, visible at the surface as white, somewhat sunken stripes, the so-called inscriptiones tendineae. Of those inscriptiones tendineae usually six can be counted [2]) limiting five muscle compartments about equal in length. We will designate the inscriptiones as I 1—6; the muscle compartments as M 1—5. The cranial and caudal part of the muscle are free from inscriptiones. In the caudal terminal part at the medial border

[1]) After research carried out in the Physiological Laboratory of the University of Amsterdam.

[2]) Those were easily found in the dogs investigated, contrarely to ELLENBERGER and BAUM who remark (Anatomie des Hundes, p. 162) „Man bemerkt an seinem Muskelbauche 3—6, jedoch nur ganz undeutliche, sehnige Inscriptionen".

often the beginning of an inscription can be found which does not penetrate the mass of the muscle at least cannot be traced further (I 7).

Peripheral Innervation.

The M. Rectus is innervated by a series of peripheral nerve branches, usually twelve in number. Of those seven go to the five compartments that are bordered by inscriptions. Of the others three spread in the unsegmented cranial part, and two in the unsegmented caudal division. Those twelve nerve branches belong to an equal number of spinal root pairs, viz. Thoracic 4 to Lumbar II.

EXPERIMENTAL PART.

The method as carried out by us consisted chiefly in stimulating the spinal roots concerned and the peripheral branches.

A. METHOD.

By opening the vertebral channel the spinal cord was exposed for at least twelve segments, between the third thoracic and third lumbar segment. In a few cases sixteen segments were exposed in order to test the entire territory that innervates the rectus muscle. In other cases a smaller number of segments caudal or cranial, was exposed. Thereupon to each root of one side (usually the left side ; twice — dog 5 and 6 — right side ; twice — dog 20 and 22 — at both sides) a conducting copper wire was fixed. Each of those wires ended in a mercury cup, numbered 1, 2 etc. and acted as unipolar stimulating electrode. A large metal plate covered with wash-leather was sewn under the skin of the neck and served as indifferent electrode. By immerging a copper rod in one of the mercury cups the secundary circuit of an inductorium could be completed and so each of the roots prepared could be stimulated at will. (Compare Fig. 1.) Now the animal was placed on

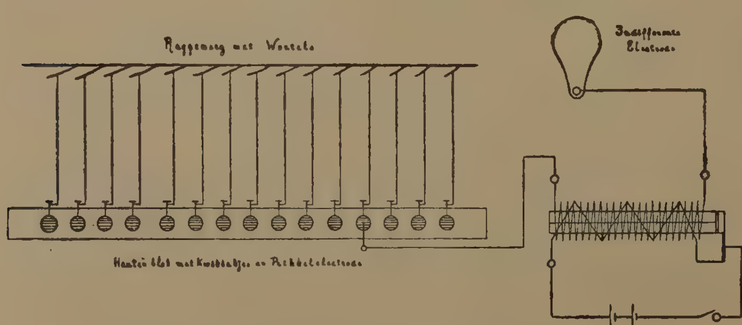

Fig. 1. Wiring diagram for stimulation of the spinal roots.

its back in order to expose the muscles of the abdominal wall. The M. rectus abdominis was freed as completely as possible from perimisial fascia, and the connections with the Mm. obliqui at the sides were carefully severed. Finally the Mm. obliqui were turned outward, exposing the M. transversus abdominis. On this muscle usually the 5 (6) most caudal peripheral branches that innervate the M. rectus take their course. The cranial branches take their origin from the intercostal nerves 4—9 or 10. Those branches cannot be exposed as easily, but it is not difficult to stimulate the same. Therefore a bipolar electrode is placed against the lower border of the rib concerned, and if necessary the ends of the electrode wire are run through the intercostal muscle in order to reach the nerve itself. The nerves that run across the M. transversus were stimulated at 2—3 cm distance from the border of the rectus muscle. The other nerve branches at a larger distance. The results of stimulation were described after simple observation and often registered photographically.

B. DESCRIPTION OF EXPERIMENTS.

Stimulation of a root partaking in the innervation of the rectus muscle always results in contraction of a small part of the muscle. Since the course of the fibres is parallel with the long axis of the muscle, the contracting part always shortens. Since the muscle is inserted into two relatively fixed points (symphysis and chest) contraction of a part of the muscle results in stretching the rest.

The contracting part increases clearly in bulk, usually the colour turns dark and the consistence becomes locally firmer. If various roots are stimulated one after another, it is very clear both visible and tangible that the contracting part changes its place.

1. *Experiments serving to determine the ventral spinal roots partaking in the innervation of the rectus muscle.*

In 15 dogs in which the caudal part of the muscle was investigated, (in two dogs at both sides) in 15 cases L II was found to be the most caudal root partaking in the innervation of the rectus muscle (Dog 1, 7. 10, 11, 13, 14, 16, 17, 18, 19, 20 and 22 R and L, and 23).

In one case the most caudal root was L III (Dog 8), and once it probably was L III (Dog 15). The inactivity of the next root was ascertained in ten out of the fifteen cases mentioned, and in the first of the exceptions.

In eight dogs the cranial part of the muscle was investigated, (two of which at both sides). In five cases the most cranial root partaking in the innervation appeared to be Th. 4 (Dog 13, 16, 19, 20, R and L), and in three dogs Th. 5 acted as such (Dog 11, 15 and 23), and in one Th. 6

(Dog 21). In three of the cases first mentioned it could be ascertained that Th. 3 was inactive.

In general a series of twelve roots, viz. Th. 4—L II appears to partake in the innervation.

2. *Experiments serving to determine the distribution of the twelve roots over the rectus muscle.*

a. The non-segmented cranial section (fig. 2).

In the innervation of the non-segmented cranial section (n-s c.s) usually three roots partake, viz. Th. 4, 5, 6. If but two roots innervate this part, stimulating Th. 5 results in contraction of the most cranial point, stimulating Th. 6 causing the caudal part to contract. (Dog 11 and 23.) In one case it was observed that Th. 6 and 7 innervated this section (Dog 21).

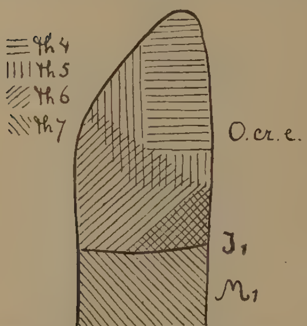

Fig. 2. Diagram of the rhizomery of the cranial non segmented section.

Sometimes Th. 7 partook in the innervation of this section by overlapping, usually supplying nerve fibres to the caudal lateral part. This happened, whether Th. 4 or Th. 5 consisted this most cranial root (Dog 13, 15, 19). In general overlapping was seen in this territory, least in the cranial lateral part, which received fibres from more than one root in one case only (Dog 19). It was a rule to find 5 and 6 overlapping, the only exception being Dog 11. Once it appeared possible that even Th. 8 had penetrated the n-s c. s.

b. The segmented medial part of the rectus muscle (fig. 3 and 4).

This medial part usually consists of five segments. Sometimes the most cranial inscription, bordering M 1 is incomplete. Some dogs showed but four segments in this part, bordered by five inscriptions. (Dog 18, 19, 21, 23).

In this territory contraction was obtained by stimulating Th. 7—Th. 13.

Th. 7. Causes as a rule contraction of M 1. In two cases (Dog 11 and 13) stimulating Th. 8 resulted also in contraction of a medial section of M 1. In Dog 20 at the left side Th. 7 left the innervation of the medial part of M 1 to Th. 8. Sometimes (Dog 13, 15, 19) Th. 7 exceeded M 1 and partook in the innervation of the n-s c.s.

Th. 8. This root has not a definite segment to innervate. In Dog 10

only M 1 was innervated independently by this root. In other cases it did not partake in the rectus innervation (Dog 7, 16) or had a small part in the innervation of M 1 or M 2 (resp. M 1 + M 2) which were innervated also by Th. 7 and 9.

Th. 9 innervates as a rule M 2 only.

Th. 10 innervates as a rule M 3 only. In one case (Dog 20) Th. 10 innervated at the right side M 3 together with Th. 11.

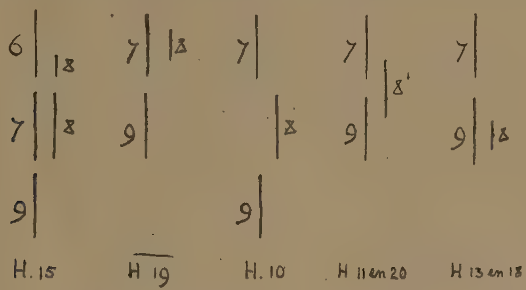

Fig. 3. Diagram of the relation of the roots Th 6, 7, 8, and 9.

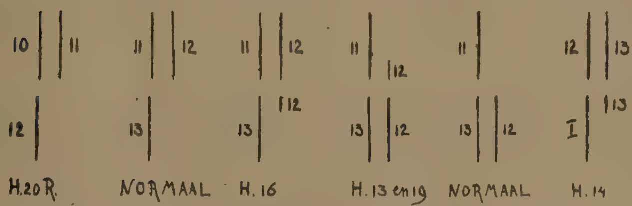

Fig. 4. Diagram of the relation of the roots Th 10, 11, 12, and 13.

Th. 11. Innervates regularly and exclusively M 4. But in half the cases Th. 12 partakes in the innervation of this segment.

Th. 12. Like Th. 8 this root does not possess a definite segment; it is true that with Th. 11 it partakes in the innervation of M 4 or with Th. 13 in M 5. Twice (Dog 13 and 19) this root innervated with Th. 13 the whole of M 5, and a small caudal part of M 4 at the same time, the remainder being innervated by Th. 11. Once (Dog 16) this root innervated the whole of M 4 together with Th. 11 and at the same time a small cranial section of M 5, the remainder being innervated by Th. 13. Twice (Dog 22 R and L) this root innervated the medial part of M 4 together with Th. 11 and the lateral part of M 5 together with Th. 13.

Th. 13. Stimulation of this root nearly always resulted in contraction of M 5. This segment may also receive fibres from Th. 12. In one case (Dog 14) stimulation of Th. 13 also caused contraction caudal to I 6, in a very small cranial part of the caudal non-segmented section of the rectus muscle.

c. The non-segmented caudal section (fig. 5).

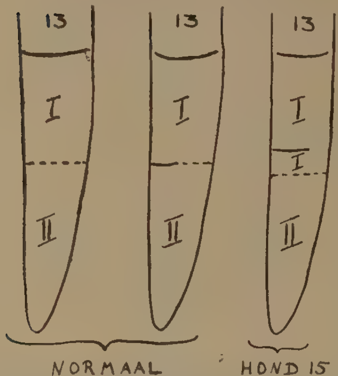

Fig. 5. Diagram of the rhizomery of the non-segmented caudal part, and of its relation to the fragmentary inscription (I 7).

The non-segmented caudal section (n-s ca.s.) as a rule is innervated by L I and L II. In one case only (Dog 8) it was L I, II and III. In one case it was uncertain whether L III partook in the innervation. Nearly always stimulating the most cranial of the roots results in contraction of a cranial section, stimulating the most caudal in contraction of the caudal part. If a larger or smaller rest of an inscription be present, the border of the two territories coincides with this inscription (I 7). Once (Dog 15) L I innervated a strip caudad to this inscription.

3. *Experiments concerning the question whether the peripheral nerve branches innervating the rectus muscle contain motor fibres from one or more ventral roots.*

In a large number of experiments we have stimulated the peripheral nerve branches and compared the effect to that obtained by stimulating the roots. Without exception it appeared that the results are identical. Stimulation of any peripheral branch innervating the rectus muscle has the same effect as stimulation of a single root. No possible interchange of fibres was observed. Each peripheral nerve branch innervating the rectus muscle consists therefore of fibres from one single root, therefore is purely uniradicular. The formation of plexus between the intercostal and lumbar nerves as often described by the anatomists, if really existing, cannot hold for the motor fibres innervating the rectus muscle.

4. *Experiments serving to discriminate more accurately the rhizomery and segmentation of the rectus muscle.*

The second chapter made it clear that at many points the territories innervated by various roots show overlapping. We want to look into this question once more.

In the non-segmented cranial section partial overlapping of Th. 5 and 6 is usual.

In the segmented medial part the territory of Th. 8 (at least if this root partakes in the innervation of the rectus muscle at all) and that of Th. 12 coincides almost always with that of the neighbouring root. For the rest each root innervates independently a single muscle segment, and respects

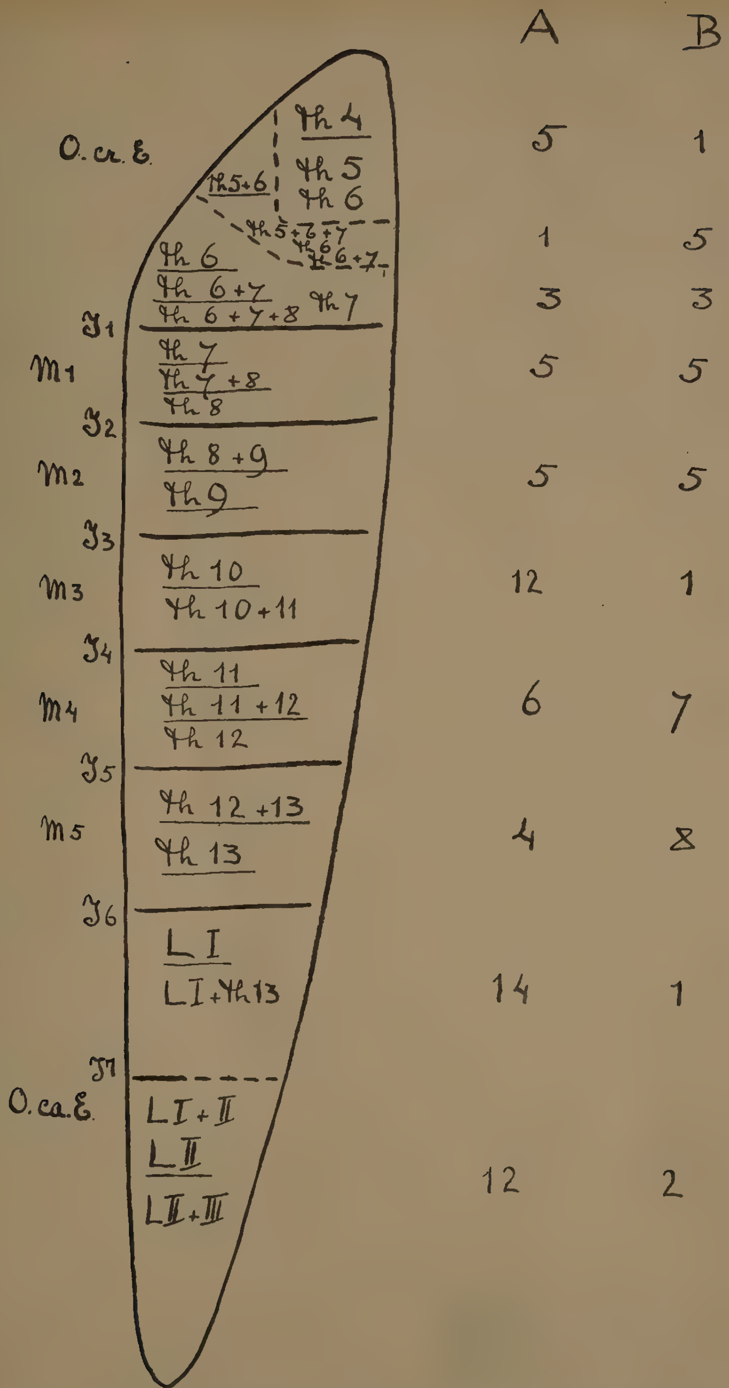

Fig. 6. Review of the relations between rhizomery and myomery in various case. Column A indicates for each segment the number of cases in which it was innervated by a single root, column B the number of cases in which fibres were supplied by two roots or more.

the segmental borders, with reservation to the exceptions as stated in chapter 2.

One remark must find here a place. The simple observation of the results of stimulating roots and nerves, even completed with photographical records is not sufficient to determine whether one or more muscle segments are in contraction. For in a dog tied on its back the insertions (high ribs and symphysis are practically fixed points, between which the muscle is stretched. If a sect of the muscle contracts the rest must be stretched over the same distance. Suppose the distribution of a root is over two segments but in a different degree, then it is possible that stimulating this root shows a result in the segment that is the most innervated, and that the small contraction in the other segment dissappears by cause af the stretching of the whole. In order to obtain an insight in this question we have measured the change in length of the rectus segments during stimulation of the roots.

TABLE 1. Dog 7. Length of segments.

Segment	In rest	Segments stimulated:						
		L II	L I	Th 13	Th 12	Th 11	Th 10	Th 9
M 2	$5^1/_2$ cm.							**$3^1/_2$**
M 3	5			6	$5^1/_2$		**$2^1/_4$**	
M 4	$4^1/_2$			$4^1/_2$	**$2^1/_2$**	**$2^1/_2$**		
M 5	$4^1/_2$			**$2^1/_4$**	**$2^1/_2$**			
N-S Ca. S.	$13^1/_2$	**10**	**10**	14	$14^1/_4$			

TABLE 2. Dog 13. Length of segments.

Segment	In rest	Segments stimulated:					
		L II	L I	Th 13	Th 12	Th 11	Th 10
N-S. Cr. S.	5 cM.						
M 1	4	4.8	4.4				
M 2	4.4	4.5	4.5				4.8
M 3	4.4	4.5	4.5	4.4		3.9	**2.4**
M 4	3.2	3.9	4	**3.1**	**2.9**	**2.2**	3.5
M 5	3.2	3.9	3.5	**3**	**2**	3.5	3 +
N-S. Ca. S. a	4.9	4.7	**3.7**	4.9	5 +	5.4	5.2
N-S. Ca. S. b	8.3	**5.6**	8.5	8.5	8.8	8.8	8.4

Those two tables show clearly that stimulation of Th. 12 causes contraction of M 4 and M 5 as well. Therefore still another method was used by us.

This method consisted in stimulating a certain root, and then the corresponding nerve. Finally this nerve was severed, and again the root stimulated.

TABLE 3.

Roots investigated:	Th 10 (D. 14)	Th 11 (D. 15, 20)	Th 12 (D. 20)
Stimulation of root causes contraction of	M 3	M 4	M 4
Stimulation of nerve causes contraction of	M 3	M 4	M 4
Stimulation after cutting nerve results in (the rectus only being considered)	nothing	nothing	nothing

Now we have the non-segmented caudal section to consider.

This part was innervated in nearly all dogs by two roots, L I and L II; in one case (Dog 8), it were L I, L II and L III.

It be premised that contraction of the entire section never was caused by stimulation of a single root. Always stimulation of one single root resulted in a partial contraction of the non-segmented section. Stimulating the cranial root always caused the cranial part to contract; stimulation of the caudal root set up contractions in the caudal part. The non-segmented section therefore consists in two well defined territories, both innervated by separate roots. Compare fig. 5 and 6.

In case the non-segmented section consisted of two territories partly divided by an incomplete inscription, the border between the rhizomers was in at least nine cases identical with it. (Dog 13, 14, 16, 17, 18, 20 and 22 R and L.) In one case (Dog 15) the border between cranial and caudal rhizomer was much more caudal than the fragmentary inscription (fig. 5). Stimulation of the nerves corresponding with the roots here also has exactly the same effect as stimulation of the roots. This is important considering the fact that the nerve from L II (N. ileo-inguinalis) is connected with a branch of L I (ELLENBERGER and BAUM). Since this connection has a more central position than the spot stimulated by us, the fact that the effect of stimulation of root and of nerve was absolutely identical proves, that this connecting branches does not contain any motor fibres from L I to the rectus muscle. Furthermore stimulation of the root of L I remained without effect after cutting the peripheral nerve branch. This also proves that no motor fibres from L I reach the rectus muscle through a way other than the segmental homonym nerve.

5. *Experiments to compare the innervation of the rectus muscle right and left in the same dog.*

In one dog we have exposed both rectus muscles and the spinal cord from Th. 4 to L III, and applied stimulating electrodes to the roots at both sides in order to ascertain the results of stimulation of roots at both sides and to compared the effects.

The following facts were elicited.

a. Concerning the structure of the muscle.

The cranial apex was at both sides at the same level (fifth rib). The cranial inscription (I 1) was at both sides incomplete and situated at the right side somewhat more craniad than at the left. At the right side I 2 was about 2 cm more craniad than at the left. I 3 showed a smaller difference, but in the same direction; I 4 and the succeeding inscriptions were situated at the same level at both sides.

b. Concerning the rhizomeric division of the muscle.

Both muscles received their innervation through a series of roots from Th. 4 to L II. In the non-segmented sections no considerable differences between the rhizomeric divisions was found. But in the segmented medial part there existed at the level of Th. 10 to Th. 12 a marked difference. In the right muscle Th. 10 and 11 together innervated M 3, and Th. 12 M 4 and no other. But in the left muscle only Th. 10 innervated M 3, Th. 12 and 11 innervating M 4. At both sides Th. 9 innervated M2 and Th. 13 M 5.

Th. 8 also showed some pecularities. Both recti showed in this case a seperate strip of muscle, stronger developed at the right side, originating from the eight rib and inserting into I 3. At the right side Th. 8 partook only in the innervation of the rectus by supplying fibres to this strip; at the left side on the other hand a medial section of M 1 was innervated also by this root.

In two other dogs (22 and 23) the innervation of the rectus muscle was similar at both sides.

CONCLUSIONS.

1. The rectus abdominis muscle in dog consists of three parts:

a. a cranial section, not divided by inscriptions, usually innervated by three spinal roots (Th. 4, 5, 6);

b. a part, divided by six inscriptions and consisting of five sections usually innervated by seven spinal roots (Th. 7—13);

c. a caudal part not divided by inscriptions, innervated by two roots (L I and II).

2. The rectus abdominis muscle of the dog as a whole does not show any segmental shifting.

3. The most caudal root partaking in the innervation nearly always is the same. In the majority of cases this is L II.

4. The segmental variations that are to be found consist of a shortening of the muscle, the non-segmented cranial section receiving fibres from two in stead of three roots, and furthermore in the lacking of a segment in the medial part which contains in those cases four in stead of five segments. The non-segmented caudal part shows the least variation.

5. In the non-segmented cranial part the rhizomers are situated one after the other, except for a slight overlapping.

In the segmented medial part M 3 coincides usually with one rhizomeric division. All other sections may correspond with two rhizomers, and therefore originate from two united myomers.

The non-segmented caudal section consists of two seperate and succeeding parts.

6. The segmental nature is not quite pure in the territory, bordered by the inscriptions and apparently very clearly segmented. On the other hand, the cranial and caudal sections, (especially the latter) not divided by inscriptions, seem to show separation of myomers and of rhizomers.

7. Th. 4 is the highest root sending fibres to the most cranial apex of the rectus muscle. Th. 5 and 6 usually innervate the succeeding parts of the non-segmented cranial section.

Th. 10 innervates nearly always only M 3, no other roots partaking in the innervation of M 3.

Th. 7 and 9 innervate M 1 and 2 ; usually Th. 8 sends motor fibres in one of the segments or in both. An independent segment was innervated by Th. 8 in one case only (Dog 10).

Th. 11 and 13 innervate almost constantly M 4 and 5. As a rule Th. 12 sends motor fibres in one of those segments or in both. In one case only (Dog 20 R) Th. 12 innervated a segment (M 4) independently.

As a rule L I innervates the cranial, L II the caudal part of the non-segmented caudal section.

Botany. — *Some remarks concerning the remains, which have been described as fossil fern-stems and petioles.* By O. POSTHUMUS. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of February 25, 1928).

In this note some remarks are made concerning the classification of the fossils, which have been mentioned as remains of fern stems and petioles, either with their internal structure still visible or preserved as impressions or medullary casts only. A number of them certainly does not belong to the Ferns; their generic names are however, summarized too, because, in the literature, they have been mentioned as fern-remains; also some forms are named here, hitherto considered to be Spermophyta, but which probably are fern stems.

In some cases these fossils may be characteristic of the strata, in which they have been found; but often they are still more valuable, because furnishing data about the habit of the stem and the internal structure of extinguished members of the Vascular Cryptogams. The most striking impression is,.that these data are widely scattered in the literature and very incomplete, but, when we realise how many interesting conclusions have already been drawn from the detailed study of the "*Inversicatenales*" by P. BERTRAND and others [1]), it becomes obvious how much may yet be expected from the rest. Especially a deeper insight into anatomical questions can be obtained only, when a more complete study of many of the fossil forms, which are mentioned in this enumeration, has been made. Not only in the interesting group of the *Dineuroidacea* and the *Clepsydropsidaceae*, characterised by the peculiar symetry of their leaves, which are grouped together by P. BERTRAND as *Inversicatenales*, as referred to above, results of fundamental interest have been obtained, but also in the *Osmundaceae* an almost continuous series of successive stages in development of their stelar structure has been found [2]). Under the names *Protopteris, Oncopteris* etc., stem-remains are known, which probably belong to the *Cyatheaceae*; slender rhizomes, named *Solenostelopteris* and *Fasciostelopteris*, are considered to belong to ferns of Polypodiaceous affinities.

Apart from better known groups, there is a number of rather obscure forms e.g. *Chelepteris, Cyatheopteris, Protopitys, Tietea*; it seems rather certain; that many of them will prove to be of utmost interest, especially in regard to their structure. When good specimens are available, a more

[1]) P. BERTRAND, Etude Zygopteridées 1909; see further; POSTHUMUS, Receuil trav. bot. néerl., XXI, 1924, p. 170—194.

[2]) POSTHUMUS, Receuil, trav. bot. néerl., XXI, 1924, p. 113—160.

detailed study will then probably prove to raise these obscure remains to the rank of material of primary importance in the arguing of fundamental questions of botany. This was already the case with the study of the fossil *Osmundaceae* by Kidston and Gwynne—Vaughan.

From the remains 99 groups of generic rank have been distinguished; they are summarised in the following list according to their affinities; the number of "species" (in total 630) which have been distinguished is indicated behind the name.

I. Stauropteridaceae.

Stauropteris Binney (2).
Bensonites R. Scott (1).

II. Dineuroidaceae.

Dineuron Renault (2).
Diplolabis Renault (3).
Etapteris P. Bertrand (6).
Arpexylon Williamson (3).
Zygopteris (s. str.) Corda (1).
Metaclepsydropsis P. Bertrand (2).
Flichea Pelourde (1).
Aphyllum Unger (1).
Androphyllum Renault (–).
Androstachys Grand'Eury (2).

III. Clepsydropsidaceae.

Clepsydropsis Unger (12).
Asteropteris Dawson (1).
Ankyropteris P. Bertrand (10).
Asterochlaena Corda (11).
Protoclepsydropsis Hirmer (1).
Botrychioxylon Scott (1).

IV. Anachoropteridaceae.

Anachoropteris Corda (5).
Calopteris Corda (1).
Chorionepteris Corda (1).

V. Botryopteridaceae.

Botryopteris Renault (17).
Tubicaulis Cotta (10).
Grammatopteris Renault (2).
Selenochlaena Corda (2).
Protothamnopteris Beck (1).
Tracheotheca Oliver (–).

VI. Psaronieae.

Psaronius Cotta (107).
Tubiculites Grand'Eury (2).
Psaroniocaulon Grand'Eury (2).
Caulopteris (s. str.) Lindley et Hutton (47).
Stemmatopteris Corda (17).
Ptychopteris Corda (14).
Stipitopteris Grand'Eury (9).
Megaphytum Artis (38).
Zippea Schimper (3).
Cromyodendron Presl (1).
Psaropteris Schimper (–).
Xylopsaronius Pohlig (1).
Gyropteris Corda pars (1).
Rothenbergia Cotta (1).
Eksdalia Kidston (1).
Ilsaephytum Weiss (3).

VII. Osmundaceae.

Thamnopteris Brongniart (4).
Zalesskya Kidston et Gwynne—Vaughan (3).
Bathypteris Eichwald (3).
Anomorrhoea Eichwald (1).
Osmundites Unger (18).
Paradoxopteris Hirmer (1).
Desmia von Eichwald (1).

VIII. Cyatheaceae.

Alsophilina Dormitzer (3).
Oncopteris Dormitzer (2).
Protocyathea O. Feistmantel (2).
Protopteris Sternberg (25).
Rhizodendron Göppert (1).
Cibotiocaulis Ogura (1).
Cyathocaulis Ogura (1).
Cyathorhachis Ogura (1).
Rhizopterodendron Göppert (1).

IX. Polypodiaceae.

Solenostelopteris Kershaw (2).
Fasciostelopteris Stopes et Fuji (1).
Tempskya Corda (12).
Sedgwickia Göppert (1).

X. GLOSSOPTERIDACEAE.

Vertebraria Royle (12).
Clasteria Dana (1).
Blechnoxylon Etheridge (1).

XI. PTERIDOPHYTA incertae sedis.

Chelepteris Corda (7).
Lesangeana Mougéot (5).
Cottaea Göppert (2).
Cyatheopteris Schimper (2).
Lepidodendrites Fliche (1).
Diplocephalis Corda (1).
Mesoneuron Unger (2).
Protopitys Göppert (2).
Psammopteris von Eichwald (1).
Selenopteris Corda (2).
Silesiopteris Posthumus (1).
Gyropteris Corda pars (1).
Sphallopteris Corda (2).
Ptilorachis Corda (1).
Stereopteris Scott et Jeffry (1).
Tietea Solms—Laubach (1).

XII. SPERMOPHYTA incertae sedis.

Arctopodium Unger (2).
Hierogramma Unger (1).
Syncardia Unger (1).
Calamopteris Unger (2).
Calamosyrinx Unger (1).
Kalymma Unger (2).
Aulacopteris Grand'Eury (3).
Steleopteris Göppert (1).
Palaeopteris H. B. Geinitz (4).
Knorripteris Potonié (2).
Adelophyton Renault (1).
Megalorhachis Unger (1).
Periastron Unger (2).
Stephanida Unger (2).

XIII. COLLECTIVE GROUP.

Caulopteris (s.l.) Lindley et Hutton (82).
Filicites Schlotheim (2).
Rachiopteris Dawson (50).
Rhizomopteris Schimper (19).
Zygopteris Corda (s.l.) (24).

The first five families were formerly not taken separate, but considered to form one group only [3]), which has been named *Inversicatenales, Coenopterideae, Botryopterideae* or *Zygopterideae*. The internal structure of their stems and petioles, however, shows such considerable differences, that they may better be divided into the above-mentioned groups.

The *Psaronieae* contain the stems of *Psaronius* showing structure and a number of stems and petioles, the internal structure of which is unknown; these are grouped into the other genera. The internal structure of Psaronius [4]) closely resembles that of some *Marattiaceae*, and it is mostly supposed that they belong to that family. It should be remarked, however, as has been done already by BOWER [5]), that the complicated structure of the vascular system is nearly as much analogous to that of some Polypodiaceae (e.g. *Pteris Kunzeana* Agardh [6]) or *Saccoloma demingense* (Sprengel) Prantl [7]) as to that of the *Marattiaceae*. Its simple leaf-trace resembles even more that of the *Polypodiaceae*; the structure of the roots, however, is more alike that of the *Marattiaceae*, but triarch and tetrarch roots have been found in *Blechnum* and *Cyathea* [8]); the filamentous interstitial tissue of the root zone has no analogy in living Ferns. From the "genus" *Caulopteris* the upper palaeozoic specimens may be included in this group.

The next three families have also representants among the living Ferns. Only from the *Osmundaceae* structural stem remains have been found in Palaeozoic strata; *Osmundites* is known to occur from the Jurassic to the Tertiary period, the other genera from Permian strata only. They show a beautiful series of successive degrees of complication of the structure of their vascular system.

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[3]) In the preface of part 12, Fossilium Catalogus, II, Plantae, an opinion is printed on p. 4, contrary to my views. We read there: „Der Name *Inversicatenales* ist nur als Sammelname zu betrachten. Es handelt sich hier eine Anzahl von Organen, von welchen mehrere zusammengehören können, oder zu anderen Gruppen, wie z.B. *Psaroniae* oder *Osmundaceae*, gerechnet werden müssen. Vorläufig jedoch müssen die einzelnen Formen noch getrennt aufgeführt werden".

In the original manuscript, however, I had written: „Der Name *Inversicatenales* ist nur ein Sammelname für einige Gruppen, welche untereinander und mit anderen Gruppen, z.B. mit den *Psaronieae* und *Osmundaceae* aequivalent sind". This opinion was in accordance with the view I had expressed before (Proc. Acad. of Science, Amsterdam 27, 1924, p. 835)· „But one should always bear in mind that it is a collective group only in wich a number of families is grouped together, the affinities of which are rather distant; and may, for instance, be compared with those of recent *Osmundaceae* and the *Gleicheniaceae*."

The discrepancy is due to the fact that I was prevented by absence to read the proofs personally, so that the editor of the Fossilium Catalogus was obliged to take care of it.

[4]) STENZEL, die Staarsteine, 1854; ZEILLER, Autun et Epinac, I, 1890, p. 178—272; RUDOLPH, die Psaronien und Marattiaceen, 1906.

[5]) BOWER, the Ferns, II, Cambridge. 1926, p. 260.

[6]) D. T. GWYNNE-VAUGHAN, Ann. of Bot., XVI, 1903, p. 702, fig. 14.

[7]) F. O. BOWER, Ann. of Bot. XXVII, 1913, p. 457.

[8]) DE BARY, Vergl. Anatomie, 1877, p. 377.

Of the *Cyatheacea* remains of fossil stems are known from the Jurassic. Cretaceous and Tertiary (*Caulopteris Laubeyi* Stenzel); their internal structure shows about the same degree of complication as in the recent species.

To the *Polypodiaceae* three genera, *Solenostelopteris* (a rhizome with a solenostele, much resembling that of a *Microlepia*), *Fasciostelopteris* (which shows some resemblance to the *Dennstaedtineae*) and *Tempskya*, are referred. The specimens of the latter genus seem to occur in Cretaceous and Tertiary strata only; their occurrence in older (Permian) strata has not been confirmed afterwards. The affinities of this genus, with its characteristic mass of roots, has not fully been made clear yet; *T. rossica* by the casual presence of two leaf-gaps, which are closer to each other than usual, may show an analogy to some species of *Cheilanthes* and *Pellaea* [9]). The stem has been compared with that of *Hemitelia crenulata* [10]).

The rhizome of *Glossopteris, Vertebraria* is also mentioned here: probably this plant, the affinities of which are not well known, belongs to the *Spermophyta*.

The eleventh group contains a number of forms, probably belonging to the *Pteridophyta*, but all of rather uncertain affinities. Some of them may, by investigation of more abundant material, prove to be of great interest. *Protopitys*, if not a group apart, shows some resemblance with *Megaphytum* and distichous species of *Psaronius*. *Sphallopteris* and *Chelepteris* possibly may belong to the *Osmundaceae*; *Tietea* is much different from any known Fern. The other forms are too badly known even to make suppositions concerning their affinities.

Another group of remains belongs to palaeozoic tribes of seedplants: *Arctopodium*, *Hierogramma* and *Syncardia* to *Cladoxylon*; *Calamopteris*, *Calamosyrinx* and *Kalymma* to *Medullosa*; *Palaeopteris* to *Cordaites*. From the other forms hardly anything can be said.

The last division contains some genera, instituted as collective groups. The Palaeozoic species of *Caulopteris* doubtless belong to the *Psaronieae*; the Mesozoic and Tertiary specimens partly to the *Cyatheaceae*, partly they are of unknown affinities. The genus *Zygopteris* has afterwards been divided by P. BERTRAND into a number of genera, grouped together by him in the *Inversicatenales*. They now form the bulk of the first five families (see note 3).

The other forms of this group are too badly known.

Concerning the nomenclature the following remarks may be made:

The combination *Botryopteris tridentata* has been made already before

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[9]) Ann. of Bot., XXVIII, 1914, p. 675 fig. 3, 7—10; fig. 4, 17—18; *(Cheilanthes)*; fig. 6, 10—19, 21—29 *(Pellaea)*.

[10]) SCHOUTE, Ann. Jard. Bot. Buitenzorg, (2) V, 1906, p. 198—207, pl. 18. 19.

by STOPES and WATSON [11]; at a former occasion [12] I had overlooked this fact.

The name *Botryopteris* for the genus of fossil plants has been given for the first time by RENAULT in 1875 [13]; in 1825 PRESL [14], however, had given already this name to a species, which appeared to belong to the genus *Helminthostachys (Ophioglossaceae)*. He used it once more for another plant in 1848 [15], this name has not been used any more by later authors. As this name had already been used for recent plants, POTONIÉ [16] distinguished the extinct genus by addition of the letter p: *p. Botryopteris;* also in *p. Callipteris* and other cases. This proposal has not been accepted by other authors.

The name *Thamnopteris* was first used by PRESL [17] for a subgenus of *Asplenium*. In 1849 he raised this group to a generic rank [18]; but most following authors continued to consider the group as a subgenus of *Asplenium* only. In 1849 BRONGNIART [19] gave the same name to a fossil stem, which afterwards appeared to belong to the *Osmundaceae*. It is not clear which of both names has been published first as a generic name; in its oldest sense, however, is not used as a generic distinction any more; the name for the fossil plants is still valid. As the rules of nomenclature for fossil plants have not yet been established, this question may be postponed until further consideration.

The name *Grammatopteris*, used at first for fossil plants by RENAULT in 1891 [20], has afterwards been used for recent ferns of the Dutch East-Indies [21]; the latter therefore have to be renamed.

In another case there is a great resemblance between a generic name of a fossil plant and a recent one. In 1818 DESFONTAINES [22] gave the name *Mezoneurum* (sometimes even written *Mezoneuron*) to a genus of the *Caesalpiniaceae*; in 1856 UNGER [23] used nearly the same name, *Mesoneuron*, to design a piece of a stem of unknown affinities, from the Upper Devonian or Lower Carboniferous of Thuringia. It will be best, also in this case, to postpone this question unto further consideration.

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[11]) STOPES AND WATSON, Trans. Roy. Soc., London, B, CC, 1908.

[12]) POSTHUMUS, Proceedings Roy. Ac., Amsterdam, 27, 1924, p. 836.

[13]) RENAULT, Ann. scienc. nat., Bot., (6) I, 1875, p. 223.

[14]) PRESL, Rel. Haenk., I, 1825, p. 76.

[15]) PRESL, Abh. kön. böhm. Ges. der Wiss., (5) V, 1848, p. 324.

[16]) POTONIE, Naturwiss. Wochenschr., XV, 1900, p. 420

[17]) PRESL, Tent. Pterid., 1836, p. 105.

[18]) PRESL, Epim. Bot., 1849, p. 68.

[19]) BRONGNIART, Tableau genres Végét. foss., 1849, p. 38.

[20]) RENAULT, Note Botryopteridées, 1891, p. 16.

[21]) v. ALDERWERELT VAN ROSENBURGH, Bull. Jard. Bot. de Buitenzorg, (3) IV, 1922, p. 318, pl. 15.

[22]) DESFONTAINES, Mém. Muséum, Paris, IV. 1818, p. 245, pl. 10, 11.

[23]) UNGER, Sandstein und Schieferflora, 1856, p. 172.

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Chemistry. — *The Reduction of α-Eleostearic Acid. (The Linoleic Acid 10.12 and the Oleic Acid 11.)* By J. BÖESEKEN and J. VAN KRIMPEN.

(Communicated at the meeting of January 28, 1928).

In a paper dealing with the α-eleostearic acid [1]) one of us in conjunction with Mr. J. HOOGLAND has communicated the preliminary results of the catalytic reduction of this acid, in which they came to the conclusion that the conjugated system of three double bonds was attached according to the principle of THIELE, so that first the octadecadeënoic 10—12 acid would be formed, and then the octadecenoic 11-acid.

For, on addition of one mol. of H_2 a compound, was obtained, which, judging from its behaviour towards the solution of WIJS, behaves as a substance with a conjugated system of double bonds, which on further reduction gave a *profuse yield* of the just-mentioned octadecenoic-11-acid (ester).

Now, however, the possibility is not excluded that on taking up the first molecule of hydrogen, one of the end-placed double bonds, either 9 or 13, was hydrogenated, so that the first obtained linoleic acid would be the octadecadeënoic 11 . 13 or the octadecadeënoic 9 . 11-acid, and that on further reduction this has given the octadecenoic-11-acid.

In order to get perfect certainty about the course of the reduction, it was, therefore, necessary, to separate the first-obtained acid, and to determine its constitution.

Besides, the octadecenoic -11-acid had not yet been obtained pure; it contained 5 % stearic acid, and the relatively high melting point 53° was possibly owing to this impurity.

Through improvement of the hydrogenation method, we have succeeded in obtaining the first reduction product of the α-eleostearic acid aethylester pure, and separating from this an acid with a conjugated system melting at 28.5°, and fixing the position of the double bonds in this acid by ozonisation at 10 and 12.

From this a good yield of *sebacic acid* and *caproic acid* was obtained, the middle part being converted to a syrup, the constitution of which we have not yet verified.

We point out here that this acid is isomeric with an acid [1]) that was obtained on distillation of ricinus-elaidic acid, and melts at 53°. This latter has been examined by Mr. W. C. SMIT, who, by determination of the refraction of the ester and from its behaviour towards the solution of

[1]) Recueil des trav. chim. **46**, 619 (1927).

WIJS, has derived that also this acid possesses a conjugated system. On ozonisation of this a good yield of *azelaic acid* and *heptoic acid* was obtained, while here too the middle two C-atoms have so far not yet yielded a definable compound in considerable quantities. The difference between this acid and that which was obtained on reduction of α-eleostearic acid is therefore confined to the position of the conjugated system, which in this latter acid lies one place further from the carbonyl group.

When the wood oil was hydrogenated with a third of the calculated quantity of hydrogen, the product, investigated according to BERTRAM's method [1]) appeared to contain no stearine. This means that this partial reduction takes place very selectively, as on one side the final product has not been reached, on the other side the about 8 % of oleine, which the wood-oil contains, have been left entirely intact.

Besides it appears from this that our initial product was free from stearic acid.

After the fixing of $^2/_3$ of the calculated quantity of hydrogen, stearine was, indeed, found, which might have been formed because the oleine was now attached, and the 11-acid is now further reduced.

As we stated above, this 11-acid separated already before was, at first, not obtained pure, it still contained 5 % stearic acid. In fact the acid once recrystallized from alcohol had a melting point of 52°—53°. If it was recrystallized from chloroform and if the first high-melting fractions were removed, the mother liquor appeared to contain a lower-melting compound, which, after repeated recrystallisations from the same solvent, melted constantly at 38°.5. We got the impression that this acid was the only component with one double bond, save the ordinary oleic acid present at the beginning.

By means of ozonisation and melting experiments the 11-oleic (elaidic) acid was found to be identical with the acid separated by S. H. BERTRAM from animal fats and called by him *vaccenic acid*. It is undoubtedly very remarkable that this is found in natural fats, and the question suggests itself whether there might be any connection between the reduction product of wood-oleic acid occurring in the vegetable kingdom, and this product of animal metabolism. The experimental details will be communicated elsewhere, but some constants way be given here.

α-Eleostearic acid. Melt. p. 45° (octodecatrienic-acid 9, 11, 13) was esterified and the aethylester was distilled in cathode-vacuum.

Boil. p. 169—170°.5, refraction at 15° of two separately obtained fractions 1.5043 and 1.5086.

HOOGLAND has also found in three fractions at 15° 1.5042, 1.5064, and 1.5080.

For the present it cannot be decided whether these relatively large differences still point to the presence of stereo isomers ; the ester is made

[1]) Chem. Weekblad 24, 226 (1927).

from α-eleostearic acid by the aid of alcohol and *hydrochloric acid*, so that an isomerisation is not excluded.

Octodecadienic acid-ethylester obtained by $^1/_3$ hydrogenation of the α-eleostearic acid ester.

Refraction: $n_D^{15°} = 1.4746$. Mol. Refr. $= 96.87$.
Calculated $= 95.28$, Exatation $= 1{,}59$ } It follows from this that there is
Iodine value $= 109$, instead of 180. } a conjugated system in this acid.
The free acid melts at 28°.5; $n_D^{70°} = 1.4639$.

This ester, as well as the wood oil itself was hydrogenated for $^2/_3$ part, the product obtained was saponified, and the acids were recrystallized, the largest fraction melted at 38°.5, showed no changes any more, yielded caproic acid and sebacic acid on ozonisation and presented no lowering of the melting-point with the *vaccenic acid* of BERTRAM (see above) $n_D^{70°} = 1.4432$.

Delft, January 1928.

Mathematics. — *A Group of Null Systems.* By Prof. JAN DE VRIES.

(Communicated at the meeting of Januari 28, 1928).

§ 1. In a null system (α, β, γ) a point has α nullplanes, a plane has β nullpoints and a line is γ times a nullray. In a paper "On Bilinear Null Systems" (These Proceedings **15**, 879) I have proved the existence of nullsystems $(1, 1, \gamma)$ for any value of γ and I have derived their properties. In a paper "Null Systems that are Defined by Two Linear Congruences of Rays" (These Proceedings **21**, 309) I have considered nullsystems $(1, pq, p+q)$ that are defined by two congruences of rays $[1, p]$ and $[1, q]$. Now I shall consider a group of nullsystems that are characterised by the symbol $(1, 2n, n+2)$.

§ 2. We shall consider as given the congruence $[k^3]$ of twisted cubics that pass through 5 points B_k and the congruence of rays $[1, n]$ formed by the lines t resting on the twisted curve α^n and on the line a which cuts α^n in $(n-1)$ points A_k.

Through a point M there pass one k^3 and one ray t; let r be the straight line that touches k^3 at M; we shall associate the plane rt to M as *nullplane* μ.

The points of contact R of the k^3 that touch a plane μ, lie on a conic ϱ [1]); its points of intersection with the n rays t in μ are the *nullpoints* of μ. Hence $\beta = 2n$.

The rays t that cut a line l, form a scroll of the degree $(n+1)$; the points of contact R of the tangents r that rest on l, lie on a cubic surface through l. Besides l the two surfaces have a curve of the order $(3n+2)$ in common that is formed by the nullpoints M of the planes μ through l. This curve cuts l in $(n+2)$ points M; hence $\gamma = n+2$ [2]).

§ 3. A point M for which r coincides with t, has a pencil of null-planes and is, therefore, *singular* for the nullsystem. I shall indicate such a point by S; let the axis of the pencil (σ) of the nullplanes be indicated by s. The lines s form a scroll; this is the intersection of the congruence $[1, n]$ and the complex of the tangents r to the curves k^3.

[1]) The plane $B_1B_2B_3$ contains a pencil (k^2) of conics each of which forms a composite k^3 together with B_4B_5. Two of these k^2 touch the plane μ.

[2]) For $n=1$ we find a null system $(1, 2, 3)$; I have treated its properties in these Proceedings **26**, 124.

As this is a complex of the degree *six* [1]), the lines s form a scroll of the degree $6(n+1)$. The *curve* λ^{3n+2} (§ 2) corresponding to l has, therefore, $6(n+1)$ points S in common with the *curve* (S) of the singular points.

In order to determine the order of (S) we shall consider the congruence $[r]$ of the tangents r that have their points of contact R in a plane φ. The points R of which the lines r meet in a point P, lie on a twisted curve of the order 7. A plane π contains two lines r; their points of contact are the points R in φ of the conics ϱ^2 in π (§ 2). The congruence $[7,2]$ of the lines r has $(2n+7)$ rays s in common with the $[1, n]$ of the rays t; accordingly the locus of the *singular points* is a curve of the *order* $(2n+7)$.

It contains the 5 base points B_k and the 10 points D each of which is the intersection of a plane $B_l B_m B_n$ and the line $B_p B_q$. Through each of these 15 points there passes one k^3 that touches a ray t. The plane $B_1 B_2 B_3$ contains four points D; they may be indicated by (12,345), (13,245), (23,145), and (45,123). Besides these four points and the three base points this plane contains $2n$ points S; they lie in pairs on the n rays t; for each of these is touched by two conics k^2 that are component parts of composite k^3.

§ 4. The nullpoints M of the planes μ that pass through a point P, lie on a surface $(P)^{n+3}$, for any ray through P is a nullray for $(n+2)$ of its points.

The surfaces $(P)^{n+3}$ and $(Q)^{n+3}$ have in common: the curve λ^{3n+2} defined by PQ, the curve $(S)^{2n+7}$, the curve α^n and the line a. Any point of α^n carries a pencil (t), hence a pencil (μ); any point of a is the vertex of a cone $(t)^n$, hence nullpoint of ∞^1 planes μ each of which contains n rays t. Accordingly a is an n-fold line and α^n is a single curve on (P). In fact $(n+3)^2 = (3n+2) + (2n+7) + n + n^2$.

The cubic surface of the conics ϱ^2 in planes through l (§ 2) has $3n$ points in common with α^n; each of these singular points has a nullplane through l. Analogously a contains three singular points that have a nullplane through l. Hence λ^{3n+2} has $3n$ *points* in common with α^n and it contains three *n-fold points* on a.

A surface $(O)^{n+3}$ has in common with λ^{3n+2}: the $2n$ nullpoints of the plane OPQ, the $6(n+1)$ points S that lie on λ (§ 3), the $3n$ points on α^n and the three n-fold points of λ. In fact $(n+3)(3n+2) = 2n + 6(n+1) + 3n + 3n^2$.

The 10 planes $B_k\, B_l\, B_m$ are *singular nullplanes*; their nullpoints lie on the n lines t in that plane; these are *singular nullrays*. Also the 10 lines $B_k\, B_l$ are *singular nullrays*, for they may be considered as lines r.

[1]) Each of the rays through a point P is cut in two points Q by a k^3. The locus (Q) is a surface of the 4th degree with conical point P. Any plane through P contains 6 tangents of (Q) that meet in P. The locus of the points of contact is a twisted curve of the order 7.

§ 5. If we replace the congruence [1,3] that has the curve α^3 and one of its chords a as directrices, by the [1,3] of the *bisecants* of α^3, we find another nullsystem (1, 6, 5) [1]). Analogous considerations lead to a curve $(S)^{13}$ that cuts the curve λ^{11} in 24 points. The curve α^3 is double on the surface $(P)^6$ and λ^{11} has 9 double points on α^3.

[1]) Also the planes of osculation of the curves k^3 form with their points of contact a nullsystem (1, 6,5). Cf. STURM, Die Lehre von den geometrischen Verwandtschaften, IV, 469.

Mathematics. — *Representation of a Bilinear Congruence of Twisted Cubics.* By Prof. JAN DE VRIES.

(Communicated at the meeting of January 28, 1928).

1. Let (α^2) be a pencil of quadratic surfaces of which the base curve consists of the curve α^3 and the line c, and (β^2) a similar pencil with basis β^3 and c. The intersection of an α^2 and a β^2 consists of a cubic ϱ^3 and the line c that cuts it twice.

Any ϱ^3 has one of its bisecants m through the fixed point M; any ray m of the sheaf about M is a bisecant of one ϱ^3; its points of intersection with ϱ^3 are the points of the common pair of the involutions in which (α^2) and (β^2) cut m. The congruence $[\varrho^3]$ is called *bilinear*, because an arbitrary point carries *one* ϱ^3 and an arbitrary line is a chord of *one* ϱ^3. As the *image* of a ϱ^3 we shall consider the point of intersection R of m and a fixed plane Φ.

The curve μ^3, that passes through M, is represented on the conic σ^2 along which the cone μ^2 that projects μ^3 out of M, cuts the plane Φ.

2. The locus of the pairs of points on the rays m is a surface μ^4 with a conical point M; the cone of contact is μ^2. The intersection of μ^4 and μ^2 consists of the curve μ^3 and a figure of the fifth order. If P is a point of this figure, MP is a bisecant of the ϱ^3 through P, but at the same time of μ^3, hence of ∞^1 curves ϱ^3. Accordingly this figure consists of five lines that are *singular bisecants.*

Two of these lines may be indicated at once. The surface α^2 through M contains a scroll to which c belongs; the line a of this scroll through M is cut by any surface β^2 in two points of a ϱ^3 of the congruence; it is, therefore, a *singular bisecant*. Analogously M carries a line b that is cut by the pencil (α^2) in an involution of which any pair belongs to a ϱ^3.

On each of the singular bisecants s_1, s_2, s_3 that also pass through M, (α^2) and (β^2) define *the same* involution. The locus of the curves ϱ^3 that have s_k as bisecant, considered as product of the pencils (α^2) and (β^2), which have become projective, is a surface Σ^4_k.

The points of intersection S_1, S_2, S_3, A and B of the singular bisecants and Φ are *singular points* for the *representation*, as each of them represents ∞^1 curves ϱ^3.

3. The system of the curves ϱ^3 resting on a line l, is represented in the points of a rational curve λ. As l contains four points of Σ^4_k, S_k is a

quadruple point of λ; analogously A and B are *double points*. With σ^2 λ has only the 5 singular points in common, as l does not generally cut the curve μ^3. Consequently λ and σ^2 have $3 \times 4 + 2 \times 2$ or 16 points in common; λ is, therefore, a curve of the *order eight*. It has a third double point in the image of the ϱ^3 that has l as bisecant.

Two curves λ^8 have $8^2 - 3 \times 4^2 - 2 \times 2^2$ or 8 non-singular points in common. Accordingly on any *two lines* there rest *eight* curves ϱ^3 and the curves that are cut by l, form a surface Λ^8.

The surface β^2 through a point of α^3 contains all ϱ^3 that the pencil (α^2) has in common with β^2. Hence Λ^8 has the base curves α^3 and β^3 as *double curves*.

Two surfaces have 8 curves ϱ^3 and the double curves α^3 and β^3 in common. Hence c is a 16-fold line of the intersection; c is, therefore, a *quadruple line* and the curves ϱ^3 that cut c in a point C, form a surface Γ^4 [1]). Γ^4 has in common with Λ^8 $4\,\varrho^3$, the curves α^3 and β^3 that must be counted twice, and the line c; this line is, accordingly, a *double line* of Γ^4.

Any curve ϱ^3 has 24 points in common with Λ^8; 8 of them lie in the points where it rests on c; the remaining 16 must lie on α^3 and β^3; consequently ϱ^3 cuts each of these base curves *four times*.

4. A surface Γ^4 has two pairs of points in common with s_k; accordingly the image curve γ of the system of curves ϱ^3 on Γ^4 has three double points S_k. Γ^4 has two points of α^3 and a pair of points of a ϱ^3 in common with the singular line a; hence γ passes through A and through B. It can only have singular points in common with σ^2 and is, therefore, a curve of the order *four*.

The curves $\gamma^4(A, B, S_k^2)$ and $\lambda^8(A^2, B^2, S_k^4)$ have four non-singular points in common; this proves again that c is cut in any of its points by four ϱ^3.

Two curves γ^4 have two non-singular points in common; through *any two* points of c there pass, therefore, *two* ϱ^3.

5. A plane through M cuts a surface α^2 along a conic which the pencil (β^2) cuts in a point C and a cubic involution. The pairs of this involution lie on the tangents of another conic; hence through M there pass two bisecants of curves ϱ^3 on α^2. Consequently the image curve of the system is a conic a^2.

a^2 can only have singular points in common with σ^2. In fact s_k and b are bisecants of ϱ^2 lying on α^2. Accordingly the *image curves* a^2 form a *pencil* with base points S_1, S_2, S_3 and B.

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[1]) This surface is produced by the projective pencils (α^2) and (β^2) in which two homologous surfaces touch each other in the point C. Hence Γ^4 has a triple point in C and is a monoid.

Analogously the curves b^2 through S_k and A represent the systems that lie on the surfaces β^2.

The fourth point of intersection of an a^2 and a b^2 is the image of the ϱ^3 that is defined by the corresponding α^2 and β^2. Especially the figures $(S_1 S_2, S_3 A)$ and $(S_1 S_2, S_3 B)$ define the *system* of a *conic* δ^2 and a *line* d cutting it.

This proves that the planes δ of the conics δ^2 form a system with index 3. Accordingly through any point of α^3, β^3 or c there pass three δ^2 and α^3, β^3 and c are triple lines on the surface $\triangle$ formed by the curves δ^2.

6. A plane through c contains one line d that rests on α^3 and β^3 and, therefore, belongs to a composite figure ϱ^3.

α^3 and β^3 are projected out of a point C by two cubic cones that have five lines d in common besides the double generatrix c. Hence the locus of the lines d is a *scroll* $(d)^6$ with *quintuple line* c containing the curves α^3 and β^3.

Through a point of α^3 there pass three δ^2 and one d; accordingly on a β^2 there lie four figures (δ^2, d). The image curve of the system of the (δ^2, d) has, therefore, quadruple points in A and in B. It has a double point in S_1, for each of the planes $S_1 S_2 M$ and $S_1 S_3 M$ contains one δ^2. Consequently the *image curve* is a $\delta^7 (A^4, B^4, S_k^2)$; for it can only have singular points in common with σ^2.

It has 16 points outside σ^2 in common with $\lambda^8 (A^2, B^2, S_k^4)$; hence the figures (δ^2, d) form a surface of the 16th degree consisting of the scroll $(d)^6$ and a $\triangle^{10}$ with *triple* lines α^3, β^3 and c.

$\delta^7 (A^4, B^4, S_k^2)$ and $\gamma^4 (A, B, S_k^2)$ show again that a monoid Γ^4 contains eight figures (δ^2, d).

The surfaces $\triangle^{10}$ and $(d)^6$ have a figure of the order 27 in common besides α^3, β^3 and c; it consists of 9 ϱ^3 degenerated in three parts. In fact α^3 and β^3 have nine bisecants in common besides c and each of these belongs to a figure consisting of *three lines*.

7. The intersection of the surface μ^4 (§ 2) and a plane through M is a c^4 with double point M; six of its tangents r meet in M.

The curves ϱ^3 that have a tangent through M, form a surface O; the image curve of this system is an r^6 with double points in A, B and S_k, for any singular bisecant is touched by two ϱ^3.

From the number of non-singular points of intersection of $r^6 (A^2, B^2, S_k^2)$ and $\gamma^4 (A, B, S_k^2)$ and $a^2 (B, S_k)$ it appears that O has the curves α^3 and β^3 as *quadruple* lines and the line c as *eightfold* line.

r^6 together with $\lambda^8 (A^2, B^2, S_k^2)$ prove that the surface O is of the *sixteenth* degree.

8. The curves ϱ^3 that touch a plane ω, form a surface Ω. In the inter-

section of an α^2 and ω the surfaces β^2 form an involution I^3; as this has 4 double points, α^2 contains four ϱ^3 of the system Ω. Hence the line a is a chord of four of the ϱ^3 and A and B are quadruple points of the image curve.

On the intersection of ω and the surface Σ_k^4 the projective pencils (α^2) and (β^2) also form a system of point triples; as the genus of this curve is two, there are in this case 8 double points. Consequently S_k is eightfold on the image curve.

This has, therefore, 32 points in common with σ^2 and is an ω^{16} (A^4, B^4, S_k^8).

It has 16 points outside σ^2 in common with λ^8 (A^2, B^2, S_k^4); accordingly the curves that touch ω, form a surface Ω^{16}. On this α^3 and β^3 are *quadruple* curves; for ω^{16} has four non-singular points in common with $a^2 (B, S_k)$.

ω^{16} and $\gamma^4 (A, B, S_k^2)$ prove together that the line c is *eightfold*.

ω^{16} and $\delta^7 (A^4, B^4, S_k^2)$ prove, that Ω^{16} contains 32 figures ϱ^3 degenerated in two parts.

Botany. — *Zur Klärung des Xerophytenproblems.* By A. SEYBOLD. (Communicated by Prof. Dr. F. A. F. C. WENT.)

(Communicated at the meeting of February 25, 1928)

Die Wasserökonomie der xeromorphen Pflanzen, die in einer Umgebung zu leben vermögen, die sich durch grossen Dampfhunger kennzeichnet, ist in den letzten Jahren häufig untersucht worden. Alle Untersuchungen stimmten in dem Ergebnis überein, dass die SCHIMPER—WARMING'sche Xerophytentheorie der eingeschränkten Transpiration gegenüber den mesophytischen Pflanzen nicht zu recht bestünde und an ihre Stelle trat die Theorie der Dürreresistenz der Xerophyten, die hauptsächlich von dem Russen MAXIMOV begründet wurde. Der Hauptinhalt der Theorie stellt sich folgendermassen kurzgefasst dar: Xeromorphe Pflanzen vermögen mittels hoher Saugkräfte mit einem gut ausgebildeten Wurzelsystem und Reduktion der Transpirationsfläche dem grossen Dampfhunger der Atmosphäre ohne Schädigung zu widerstehen, eine Einschränkung der Transpiration pro Flächeneinheit weisen sie den Mesophyten gegenüber nicht auf, vielmehr kann ihre Transpiration flächenrelativ höher sein.

Demnach müsste die Xeromorphie mit anderen als transpirationsphysiologischen Verhältnissen in Zusammenhang gebracht werden. Die anatomisch-histologischen Strukturen xeromorpher Blätter hätten für die Wasserdampfabgabe keine Bedeutung, was allein vom physikalischen Standpunkte aus sehr sonderbar erscheint.

Im Rahmen einer eingehenden Transpirationsanalyse der Pflanze auf physikalischer Grundlage ergaben sich für das Xerophytenproblem neue Perspektiven, die zur Klärung dieser strittigen Frage beitragen dürften.

Die scharfe Scheidung, die SACHS für die Transpirationsanalyse macht: Die Transpiration aus Zellen und Geweben wird durch äussere und durch innere Ursachen und Bedingungen hervorgerufen und verändert, erweist sich auch hier äusserst fruchtbar. Vergleiche zwischen Meso- und Xerophyten können nur unter denselben äusseren Bedingungen einwandfrei angestellt werden. Der Oekologe, der die Pflanzen am natürlichen Standorte beobachtet, darf sich über diese fundamentale Forderung ebenso wenig hinwegsetzen wie der Laboratoriumsphysiologe. Dass eine xeromorphe Pflanze in dampfdruckarmer Luft flächenrelativ mehr transpirieren kann als eine mesophytische in ± stark dampfgesättigter, ist selbstverständlich. Dass aber die xeromorphen Blattstrukturen allein einem durch das xerophytische Klima physikalisch bedingtem Wasserdampfaustausch ohne Schädigung nachkommen können, wird kaum von der Hand zu weisen sein. Vom

physikalischen Standpunkte aus muss eine Verdickung der Kutikula, ein starker Wachsüberzug, Haarbildungen u.s.w. die kutikuläre Transpiration erniederigen. Durch die Ausbildung solcher histologischer Elemente kommen aber die Stomata häufig in Lagen, denen ein relativ hoher Dampfdruck eigen sein muss. Das gilt für alle mehr oder weniger tief eingesenkten Stomata.

Werden Mesophyten mit Xerophyten verglichen, so muss das mit anderen Worten heissen: Wie reagieren die Transpirationssysteme der Mesophyten und die der Xerophyten auf mesophytisches und wie auf xerophytisches Klima? Dieser Umstand ist völlig ausser Acht geblieben und mit einer Umrechnung auf dasselbe Sättigungsdefizit ist ein einwandfreier Vergleich keineswegs statthaft, wie es beispielshalber MAXIMOV in seiner grundlegenden Arbeit getan hat. Die Verdunstung ist nicht schlechthin einer der äusseren Bedingungen (Temperatur, oder falscherweise dem Dampfdruckdefizit) proportional, sondern sie als mehrgliederige Funktion der physikalischen Zustände zu betrachten.

Pflanzen, die in xerophytischem Klima zu vegetieren vermögen, was ihnen durch xeromorphe Strukturen möglich ist, sind demnach prinzipiell von den Mesophyten unterschieden, die in einem ausgesprochen xerophytischen Klima nicht zu leben vermögen. Die mesophytische Pflanze hat eine relativ sehr hohe Kutikulartranspiration dem Xerophyten gegenüber und schon aus diesem Grunde wäre ihre Transpiration in xerophytischem Klima ungleich höher, wenn sie den starken Wasserverlust der Dürreresistenz decken könnte, was eine Frage der Leistungsfähigkeit des Wurzel- und Wasserleitungssystems ist.

Hier möge nur ein klimatischer Faktor in Betracht gezogen werden, der zweifelsohne eine der wichtigsten Funktionen der Wasserdampfbewegung bei den Xerophyten am natürlichen Standorte ist, die Luftbewegung und zwar mit geringem Dampfdruck, also landläufig gesprochen: trockener Wind. Da an anderer Stelle ein gut fundierter Beweis gegeben werden wird, dass die kutikuläre Transpiration in erster Linie durch den Wind eine Steigerung erfährt, den Gesetzmässigkeiten der Verdampfung relativ grosser Flächen folgend, begnügen wir uns hier mit den physiologischen Tatsachen, die aus einer Reihe von Experimenten gewonnen wurden. Vermöge der ungewöhnlich starken Kutikula ist die kutikuläre Transpiration gleich Null zu setzen, die Verdunstung findet also lediglich durch die Stomata statt. Nun ist aber von der grössten Bedeutung zu erfahren, dass die stomatäre Transpiration der Xerophyten durch den Wind überhaupt keine Steigerung erfährt, die täglichperiodische Spaltenapertur wird durch den Wind in keiner Weise beeinträchtigt, was für die Beurteilung des Gesamt-Gas-Austausches von Bedeutung ist. Die Assimilation erfährt somit durch eine Spaltenverengung, die bei starkem Wind eintreten könnte, keine Hemmung, da der CO_2 Diffusion keine Widerstandserhöhung auferlegt wird. Anders dagegen

liegen die Verhältnisse bei den Mesophyten. Starker Wasserverlust bedingt Turgorerniederigung in den Schliesszellen der Stomata, d.h. die Spalten erfahren eine Verkleinerung. In zweierlei Hinsicht ist also das Verhalten der xeromorphen Systeme von Bedeutung. Die Transpiration wird im W i n d e nicht gesteigert, und die Spaltenapertur erleidet keine Veränderung. Damit wird die CO_2-Diffusion nicht gehemmt, rein der relativen Grösse der Gesamtporenfläche gemäss. Die Mesophyten aber, die im Winde ± rasch einen Spaltenschluss durch Turgorerniederigung eintreten lassen, um allzugrossem, schädlichem Wasserverlust vorzubeugen, erhöhen damit den Diffusionswiderstand für CO_2 im Sinne der Flächenverkleinerung. Damit ist eine physiologische Erklärung möglich, dass die Xerophyten in xerophytischem Klima zu leben vermögen, nicht aber die Mesophyten. Es frägt sich dann nur noch warum im allgemeinen die Xerophyten nicht ebenso häufig in mesophytischem Klima leben. Diese Frage nur nach dem Stande der Wasserbilanz zu beurteilen, wäre hinsichtlich der grossen Zahl der determinierenden Faktoren zu gewagt, doch können folgende Momente mit in Rechnung gesetzt werden. Die Geschwindigkeit des Wasserstromes wird bei Xerophyten im mesophytischen Klima, dem ein relativ hoher Dampfdruck eigen ist, verzögert, die Stoffwechselvorgänge werden also in erster Linie in Mitleidenschaft gezogen, die Wachstumsgeschwindigkeit aber korrelativ benachteiligt. Die klimatischen Faktoren wirken ohne Zweifel selektiv auf die zur Keimung kommenden, zufälligen Samenaggregate; mesophytische Pflanzen wachsen im mesophytischen Klima rascher gross als xerophytische. Damit soll keineswegs geleugnet werden, dass Xerophyten ganz und gar nicht in mesophytischem Klima leben könnten, ebensowenig, dass eine Pflanzenart nicht mit Standortsmodifikationen auf äussere Induktionen sich zu ändern vermöchte. Grundlegend erscheint uns aber die grosse Wirkung der Transpirationsforderung des t r o c k e n e n W i n d e s bei mesophytischen Strukturen, seine Inaktivität bei der Wasserbilanz der Xerophyten.

Die beigefügten Tabellen geben kontinuierliche Gewichtsverluste als Ausdruck des Transpirationsstromes in bestimmter Zeit wieder. Hier sei von relativen Berechnungen verschiedener Pflanzen abgesehen, da es nur darauf ankommt zu zeigen, dass die Transpiration im Winde bei dem xeromorphen N e r i u m O l e a n d e r keine Steigerung erfährt, wohl aber bei D a t u r a s u a v e o l e n s, die typisch mesomorph ist, infolge starker Kutikulartranspiration. Die physikalischen Aussenbedingungen sind während eines Versuches nahezu gleich, die Angaben sind den Tabellen beigefügt. Um aber wirklich unter sich vergleichbare Werte zu bekommen, sind die unter Wasser abgeschnittenen Sprosse, die in Wassergefässen abgedichtet standen, zum Teil zu gleicher Zeit dem Winde ausgesetzt worden, zum andern denselben Aussenbedingungen, dabei aber in ruhiger Luft verweilend. Intermittierend standen die Pflanzen in Wind und Ruhe. Die Gewichtsverluste die im Winde eintraten sind in den Tabellen fettgedruckt. Die Ursachen der Schwankungen sollen hier nicht diskutiert werden,

Nerium Oleander.

Mittlere Temperatur 30°. Mittl. rel. Feuchtigk. 25 %. 2 Tageslichtlampen 500 Watt/220V.

Zeit	Pflanze 1	2	3	4	5
Gewichtsverlust in mg von:	Windgeschwindigkeit 1.7 m/sec.				
12—13h	**0.690**	**0.340**	0.510	0.765	0.440
13—14	**0.255**	**0.240**	0.255	0.280	0.230
14—15	0.215	0.210	**0.305**	**0.190**	0.230
15—16	0.290	0.370	**0.305**	**0.125**	0.260
16—17	**0.260**	0.370	0.345	0.135	**0.210**
	Windgeschwindigkeit 5.6 m/sec.				
17—18	**0.250**	0.380	0.295	0.160	**0.290**
18—19	0.280	0.290	0.330	0.150	0.345
19—22 Mittelw. pro Stunde	**0 303**	**0.247**	0.317	0.210	0.330
22—8 Mittelw. pro Stunde	0.137	0.087	**0.182**	0.245	**0.139**

Datura Suaveolens.

Mittlere Temperatur 23°. Mittlere Feuchtigk. 40 %.

Zeit	Pflanze 1	2	3	4	
	mg Gewichtsverlust				
12^{05}—12^{20}	0.270	0.190	0.145	0.080	
12^{20}—12^{35}	**0.600**	**0.370**	0.140	0.090	
12^{35}—12^{50}	0.290	0.190	**0.340**	**0.160**	
12^{50}—13^{05}	0.420	0.240	0.230	0.150	
13^{05}—13^{20}	**0.550**	**0.310**	0.220	0.140	Windges. 1.7 m/sec. Sonnenlicht
13^{20}—13^{35}	0.260	0.180	**0.400**	**0.210**	
13^{35}—13^{50}	**0.310**	**0.180**	0.280	0.110	
13^{50}—14^{05}	0.140	0.110	**0.340**	**0.160**	
14^{05}—14^{20}	0.150	0.110	0.170	0.110	
14^{20}—14^{35}	0.170	0.140	0.210	0.130	

die Werte veranschaulichen deutlich, dass die Transpiration bei Nerium vom Winde nicht beeinflusst wird, der Gang der Transpiration ist von ihm völlig unabhängig. Anders bei Datura. Im Winde wird die Verdunstung ± stark gefördert, wenngleich auch andere Faktoren bei der absoluten Grösse der Transpiration eine nicht untergeordnete Rolle spielen.

Absolute Vergleiche beider Pflanzen sind ohne umständliche Berechnungen nicht möglich. Um dies zu umgehen werden bei künftigen Untersuchungen, die vor allem die Maxima der Transpirationsleistungen bei verschiedenen Pflanzentypen klarlegen sollen, Meso- und Xerophyten nebeneinander im Experimente behandelt, soweit dies bei extremen Aussenbedingungen mit Mesophyten sich verwirklichen lässt. Derartige Versuche sind bereits in Angriff genommen, worüber später zu berichten sein wird.

Utrecht, Februari 1928.